Sums and Products

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Introduction

Let us start with a simple (?) question. In a chess tournament there are n players who play eliminatory games until only one winner remains. Thus, in the first round, the players are arbitrarily playing games (chosen by drawing lots), and only the winners of these games go to the second round. If the number of players is odd there is one player staying aside, but he/she (and all winners) will take part to the drawing for the second round. The process repeats in the second and all the following rounds until, as we said, finally, only one winner remains (and he is declared the winner of the championship). The question is: how many games are necessary in order to establish the champion?

Well, you might need a moment of thinking, and we strongly advise you to take it (or, maybe, you already got the answer, which is great). You will immediately see that if $n = 2^m$ is a power of 2, then in the first round there are 2^{m-1} games (and 2^{m-1} winners from these games accede to the second round), and the process goes on and on so that there will be 2^{m-j} games in the jth round. The total number of games will then be

$$2^{m-1} + 2^{m-2} + \cdots + 2 + 1 = 2^m - 1 = n - 1.$$

Although we cannot quite use this reasoning in the general case, the answer $n-1$ is correct for each and every value of n, because in each game precisely one player is eliminated and, in order to arrive to the situation when only one player still stands, $n-1$ players must be eliminated, so, $n-1$ games are needed to see who the champion is.

The problem is solved, but we won't stop here. This is because the reasoning in the particular case of $n = 2^m$ furnishes a hint for the general case. Namely,

in the first round the number of games is k if $n = 2k$ is an even number, and it is also k if $n = 2k + 1$ is odd (and a player is forced – also by drawing, to make the competition fair – to stay aside). This number can be expressed (for both cases of even and odd n) as $\lfloor n/2 \rfloor$ where $\lfloor x \rfloor$ denotes the integer part of x (or the floor function of x) – the largest integer which is not greater than x (that is, $\lfloor x \rfloor = p$ if and only if p is the only integer such that $p \le x < p + 1$). Thus in the first round there are $\lfloor n/2 \rfloor$ games and in the second round

$$n_2 = n - \left\lfloor \frac{n}{2} \right\rfloor$$

players participate. Then the same pattern repeats: $\lfloor n_2/2 \rfloor$ games are played in the second round, and

$$n_3 = n_2 - \left\lfloor \frac{n_2}{2} \right\rfloor$$

players enter the third round, and so on. Thus, if we define the function f by

$$f(x) = x - \left\lfloor \frac{x}{2} \right\rfloor$$

the total number of games played is

$$\left\lfloor \frac{n_1}{2} \right\rfloor + \left\lfloor \frac{n_2}{2} \right\rfloor + \cdots$$

where the sequence $(n_k)_{k\ge1}$ is defined by $n_1 = n$ and the recurrence $n_k = f(n_{k-1})$ for every $k \ge 2$. One can easily see that, starting with every positive integer n the terms of the sequence $(n_k)_{k\ge1}$ eventually become equal to 1, because this is a sequence of positive integers that strictly decreases as long as its terms are greater than 1 (and thus, at some moment, a term equals 1, and then all the terms that follow are also equal to 1). Thus the above sum is actually a finite one (as we have already seen in the particular case of n being a power of 2), as the integer parts $\lfloor n_k/2 \rfloor$ are 0 as soon as n_k becomes 1. Actually, we can find a formula for n_k, namely

$$n_k = \left\lceil \frac{n}{2^{k-1}} \right\rceil,$$

where the ceiling function of the real number x is defined by $\lceil x \rceil = q$ if and only if q is the unique integer such that $q - 1 < x \le q$, and this formula

allows us to see that n_k becomes 1 as soon as $2^{k-1} \geq n$ (and the first such k is $\lceil \log_2 n \rceil + 1$) – but this is not our point of interest here. Nevertheless, we are interested in the fact that

$$\left\lfloor \frac{n_1}{2} \right\rfloor + \left\lfloor \frac{n_2}{2} \right\rfloor + \cdots = n - 1,$$

since we know now that the total number of games is $n-1$. For instance, with $n_1 = n = 7$, we have

$$\begin{aligned} n_2 &= n_1 - \left\lfloor \frac{n_1}{2} \right\rfloor = 7 - 3 = 4, \\ n_3 &= n_2 - \left\lfloor \frac{n_2}{2} \right\rfloor = 4 - 2 = 2, \\ n_4 &= n_3 - \left\lfloor \frac{n_3}{2} \right\rfloor = 2 - 1 = 1, \end{aligned}$$

and, consequently, $n_k = 1$ for all $k \geq 4$. Thus the number of games measured as a sum is

$$\left\lfloor \frac{7}{2} \right\rfloor + \left\lfloor \frac{4}{2} \right\rfloor + \left\lfloor \frac{2}{2} \right\rfloor + \left\lfloor \frac{1}{2} \right\rfloor + \cdots = \left\lfloor \frac{7}{2} \right\rfloor + \left\lfloor \frac{4}{2} \right\rfloor + \left\lfloor \frac{2}{2} \right\rfloor.$$

Of course, this is $3 + 2 + 1 = 6$ and corresponds to the result $7 - 1$ given by the (let us call it) global reasoning from the beginning.
To summarize: we defined a function

$$f(x) = x - \left\lfloor \frac{x}{2} \right\rfloor$$

and a sequence $(n_k)_{k \geq 1}$ starting with some arbitrary positive integer $n_1 = n$ and satisfying the recurrence relation $n_k = f(n_{k-1})$ for $k \geq 2$, and we obtained

$$\left\lfloor \frac{n_1}{2} \right\rfloor + \left\lfloor \frac{n_2}{2} \right\rfloor + \cdots = n - 1.$$

This may be a somehow unexpected equation (although it is very clear in the particular case $n = 2^m$) and illustrates a simple principle in mathematics: do something in two different ways (in this case "do something" is "count"), then equate the two results (they must be equal, because, in the end, they represent the same thing – in our case the total number of games). You will be amazed

of what can be obtained by using this very simple (because fundamental) principle.

Anyway, it is not our purpose here to examine such reasonings. Instead, we are dealing throughout this book with sums and products (as the title says). Sums and products are everywhere in mathematics. Probably the first matters that a man (usually a child) learns in mathematics are addition and multiplication (of natural numbers, then of integers, and so on) – operations for which the results are called sum and product respectively. Clearly, we do not intend to take it all from zero, but rather we will try to calculate as many sums as possible of the form

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n$$

and products of the form

$$\prod_{k=1}^{n} a_k = a_1 a_2 \cdots a_n$$

(that is, sums and products of an arbitrary number of terms, respectively factors). Also, we will enter a little in the more advanced topic of infinite sums and products, defined as limits of corresponding finite partial sums and products, respectively:

$$\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} \sum_{k=1}^{n} a_k$$

and

$$\prod_{k=1}^{\infty} a_k = \lim_{n\to\infty} \prod_{k=1}^{n} a_k.$$

We assume the reader to be familiar with the basic concepts of limit (and to have the knowledge of elementary limits), and (more rarely) of derivative and Riemann integral. There are, however, only few examples of this kind, and the reader who is not familiar with these concepts can skip them without losing the rest of the book. This clearly means that we expect the reader to know basic arithmetic, algebra, and trigonometry (complex numbers in algebraic

and trigonometric form included). Also, some combinatorial problems, and a few problems of number theory will be encountered.
Thus, we hope the reader understands a few basic properties of the symbols defined above, such as

$$\sum_{k=1}^{n}(a_k+b_k)=\sum_{k=1}^{n}a_k+\sum_{k=1}^{n}b_k,$$

or, more generally,

$$\sum_{k=1}^{n}(\alpha a_k+\beta b_k)=\alpha\sum_{k=1}^{n}a_k+\beta\sum_{k=1}^{n}b_k,$$

where all of α, β, a_k, and b_k are (in general) complex numbers, with α and β being, of course, independent of k – the so called index of summation. By the way, this index can (in the same problem, or along the same computation) be denoted by different letters; thus

$$\sum_{k=1}^{n}a_k=\sum_{l=1}^{n}a_l \quad\text{or}\quad \prod_{i=1}^{n}a_i=\prod_{j=1}^{n}a_j.$$

We also have

$$\sum_{k=1}^{n}a_k=\sum_{k=1}^{m}a_k+\sum_{k=m+1}^{n}a_k \quad\text{and}\quad \prod_{k=1}^{n}a_k=\prod_{k=1}^{m}a_k\cdot\prod_{k=m+1}^{n}a_k$$

for $1\le m\le n$. Maybe wording this as

$$\sum_{k=1}^{n}a_k=\left(\sum_{k=1}^{m}a_k\right)+\left(\sum_{k=m+1}^{n}a_k\right) \quad\text{and}\quad \prod_{k=1}^{n}a_k=\left(\prod_{k=1}^{m}a_k\right)\cdot\left(\prod_{k=m+1}^{n}a_k\right)$$

would be more accurate, but we prefer the first form, apart from the situation when we desperately need to avoid confusion.
All the above are clear consequences of the properties of addition and multiplication (commutativity, associativity, distributivity of multiplication over

addition). The same is true for

$$\sum_{k=1}^{n}(b_{k+1}-b_k)=b_{n+1}-b_1 \quad \text{and} \quad \prod_{k=1}^{n}\frac{b_{k+1}}{b_k}=\frac{b_{n+1}}{b_1}.$$

Understanding these equalities and learning to work with them (or with similar ones) is very important because they represent a powerful tool for evaluating sums and products with an arbitrary number of terms. More precisely, when we have to calculate a sum

$$\sum_{k=1}^{n} a_k,$$

expressing the general term a_k in the form

$$a_k = b_{k+1} - b_k$$

is very effective, due to the above formula: the numerous cancellations allow us to find a simple closed formula for the given sum. We call such a sum *telescopic* (or we say that the sum telescopes, etc.). This is because we can write

$$\sum_{k=1}^{n} a_k = \sum_{k=1}^{n}(b_{k+1}-b_k) = -b_1 - b_2 - \cdots - b_n + b_2 + \cdots + b_n + b_{n+1} = b_{n+1} - b_1.$$

One of the simplest examples is the one that appeared in the beginning, namely

$$1 + 2 + \cdots + 2^{n-1} = \sum_{k=1}^{n} 2^{k-1}.$$

Thus we have $a_k = 2^{k-1} = 2^k - 2^{k-1} = b_{k+1} - b_k$ for $k = 1, 2, \ldots, n$, and we can consider $b_k = 2^{k-1}$ (the b_k appear here to be equal to the a_k; of course, this does not usually happen). Consequently,

$$\begin{aligned}
1 + 2 + \cdots + 2^{n-1} &= \sum_{k=1}^{n} 2^{k-1} = \sum_{k=1}^{n}(2^k - 2^{k-1}) \\
&= \sum_{k=1}^{n}(b_{k+1} - b_k) = b_{n+1} - b_n \\
&= 2^n - 1.
\end{aligned}$$

Throughout the book we will simply write this as

$$\sum_{k=1}^{n} 2^{k-1} = \sum_{k=1}^{n}(2^k - 2^{k-1}) = 2^n - 1.$$

As we said, we expect the reader to be familiar with some simple computations (and we think that this first sum that we just evaluated does not represent a mystery for our readers; we just used it as an example). Also, we hope that the fact that we replaced m by n does not represent an issue (the formula $\sum\limits_{k=1}^{m} 2^{k-1} = 2^m - 1$ is the same as $\sum\limits_{k=1}^{n} 2^{k-1} = 2^n - 1$, isn't it?). We preferred n to be more in the vein of what we just discussed about telescoping sums. Yet another simple example is the sum of the first n odd positive integers, that is

$$1 + 3 + \cdots + (2n-1), \quad \text{or} \quad \sum_{k=1}^{n}(2k-1).$$

Can you see the telescope? We have

$$\sum_{k=1}^{n}(2k-1) = \sum_{k=1}^{n}(k^2 - (k-1)^2) = n^2 - 0^2 = n^2,$$

yielding a beautiful formula: the sum of the first n odd positive integers equals the square of n. In order to do this computation you only need to know the elementary algebraic formulae

$$(a \pm b)^2 = a^2 \pm 2ab + b^2.$$

More specific, we need $(k-1)^2 = k^2 - 2k + 1$, but we use it in the form

$$2k - 1 = k^2 - (k-1)^2.$$

This is the main difficulty when we try to evaluate a sum (or a product) by the telescoping method: how to find the numbers b_k? Of course, this depends on the skills and the experience of each solver. If one can find a closed form for

a sum, then it is always possible to evaluate that sum by telescoping. Indeed, if we have

$$\sum_{k=1}^{n} a_k = S_n$$

for any positive integer n, then we also have

$$a_n = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n-1} a_k = S_n - S_{n-1}$$

for any n (where we define $S_0 = 0$). Thus we have $a_k = b_{k+1} - b_k$ for $b_k = S_{k-1}$. So, if we have the result, we can also telescope (but we prefer to be able to find the telescope ourselves). For instance, we have the (again well-known) identity

$$\sum_{k=1}^{n} k = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}.$$

It is often said that, asked – when he was a little boy – by his teacher to sum the first 100 positive integers, Gauss did the job immediately, to the great surprise of the teacher, who had no idea about this method. His approach uses again fundamental properties of addition. First, we have

$$\sum_{k=1}^{n} a_k = \sum_{k=1}^{n} a_{n-k+1}$$

(since addition is commutative, we can reverse the order of summation), then

$$\sum_{k=1}^{n} a_k = \frac{1}{2}\left(\sum_{k=1}^{n} a_k + \sum_{k=1}^{n} a_{n-k+1}\right) = \sum_{k=1}^{n} \frac{1}{2}(a_k + a_{n-k+1})$$

(because if $A = B$, then $A = (A+B)/2$, too). In our case,

$$\sum_{k=1}^{n} k = \sum_{k=1}^{n} \frac{1}{2}(k + n - k + 1) = \sum_{k=1}^{n} \frac{n+1}{2} = \frac{n(n+1)}{2}.$$

(Note that, in general, $\sum_{k=1}^{n} a = na$ when a does not depend on the summation index; for example, $\sum_{k=1}^{n} 1 = n$.)

Basically, Gauss observed that the sums of the kth term and $(n-k+1)$th term of the given sum are all equal (to $n+1$), and he paired terms having the same sum. This can be done in general for an arithmetic progression $(a_n)_{n\geq 1}$ (that is, a sequence for which the differences $a_{k+1} - a_k$ are all equal to the same number d, called the common difference of the progression) in order to get the sum of the first n terms; as above, we have

$$\sum_{k=1}^{n} a_k = \sum_{k=1}^{n} \frac{1}{2}(a_k + a_{n-k+1}) = \frac{n(a_1 + a_n)}{2} = \frac{n(2a_1 + (n-1)d)}{2},$$

because $a_k + a_{n-k+1} = a_1 + a_n$ for every $k = 1, 2, \ldots, n$. Note that if we use the formula $a_k = a_1 + (k-1)d$ for the general term of the progression (with d the common difference) and the above formula for the sum of the first n (actually, here, the first $n-1$) positive integers, we can also evaluate this sum as

$$\sum_{k=1}^{n} a_k = \sum_{k=1}^{n}(a_1 + (k-1)d) = \sum_{k=1}^{n} a_1 + d\sum_{k=1}^{n}(k-1) = na_1 + \frac{(n-1)n}{2}d.$$

Going back to Gauss's sum, now that we have the formula, we can also prove it by mathematical induction. To verify it for $n = 1$ is immediate, so we still need to show that if it is true for n, then it also works for $n+1$. Indeed, if

$$\sum_{k=1}^{n} k = 1 + 2 + \cdots + n = \frac{n(n+1)}{2},$$

then

$$\begin{aligned}\sum_{k=1}^{n+1} k &= \sum_{k=1}^{n} k + (n+1) = (1 + 2 + \cdots + n) + (n+1) \\ &= \frac{n(n+1)}{2} + (n+1) = \frac{(n+1)(n+2)}{2}.\end{aligned}$$

Also, we can prove it by telescoping:

$$\sum_{k=1}^{n} k = \sum_{k=1}^{n} \left(\frac{k(k+1)}{2} - \frac{(k-1)k}{2} \right) = \frac{n(n+1)}{2}.$$

Another famous example is the sum of a geometric progression

$$S = 1 + x + x^2 + \cdots + x^N = \sum_{n=0}^{N} x^n.$$

If $x \neq 1$, we can find a telescope by first multiplying through by $1 - x$ giving

$$(1-x)S = \sum_{n=0}^{N} (1-x)x^n = \sum_{n=0}^{N} (x^n - x^{n+1}) = 1 - x^{N+1},$$

and hence

$$S = \frac{1 - x^{N+1}}{1 - x}.$$

(Of course we could have also telescoped by noting that $x^n = \frac{x^n}{1-x} - \frac{x^{n+1}}{1-x}$.) A few other simple telescoping sums include (think for yourself before reading the solution)

$$\sum_{k=1}^{n} k \cdot k! = \sum_{k=1}^{n} ((k+1)! - k!) = (n+1)! - 1,$$

or

$$\begin{aligned}
\sum_{k=1}^{n} k^3 = \sum_{k=1}^{n} k^2 \cdot k &= \sum_{k=1}^{n} k^2 \cdot \frac{(k+1)^2 - (k-1)^2}{4} \\
&= \sum_{k=1}^{n} \left(\left(\frac{k(k+1)}{2} \right)^2 - \left(\frac{(k-1)k}{2} \right)^2 \right) \\
&= \left(\frac{n(n+1)}{2} \right)^2.
\end{aligned}$$

(Notice the beautiful result

$$\sum_{k=1}^{n} k^3 = \left(\sum_{k=1}^{n} k\right)^2 \Leftrightarrow 1^3 + 2^3 + \cdots + n^3 = (1 + 2 + \cdots + n)^2,$$

which is a rarity in the world of sums of powers of the first n integers.)
Also we have

$$\sum_{k+1}^{n} k(k+1) = \sum_{k=1}^{n} \left(\frac{k(k+1)(k+2)}{3} - \frac{(k-1)k(k+1)}{3}\right) = \frac{n(n+1)(n+2)}{3},$$

which permits us to evaluate

$$\begin{aligned}\sum_{k=1}^{n} k^2 &= \sum_{k=1}^{n} k(k+1) - \sum_{k=1}^{n} k \\ &= \frac{n(n+1)(n+2)}{3} - \frac{n(n+1)}{2} \\ &= \frac{n(n+1)(2n+1)}{6}.\end{aligned}$$

It would be hard to notice that this sum can be telescoped by using

$$k^2 = \frac{k(k+1)(2k+1)}{6} - \frac{(k-1)k(2k-1)}{6},$$

wouldn't it?
Now let us find a closed form for the (very important, as we will see) sum

$$S_n = \sum_{k=0}^{n} \binom{n}{k} x^k$$

where x is an arbitrary number, and the *binomial coefficients* $\binom{n}{k}$ are defined by

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)\cdots(n-k+1)}{k!}$$

for integers $n \geq k \geq 0$. One also calls $\binom{n}{k}$ "n choose k" because this number counts all possibilities of arbitrarily choosing k objects from n given objects, disregarding their order. The equality

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

can be immediately verified for integers n and k with $1 \leq k \leq n-1$ by direct computation. It is called the recursive formula of the binomial coefficients. Sometimes we will use the convention $\binom{n}{k} = 0$ for $k > n$, or for $k < 0$; with this in mind the recursive formula holds for $k = 0$ and for $k = n$, too.
Using the equality $0! = 1$ (again, a convention) one finds immediately $S_0 = 1$ (and $S_1 = 1 + x$, and $S_2 = 1 + 2x + x^2 = (1+x)^2$). Then we have, for $n \geq 2$,

$$\begin{aligned} S_n - S_{n-1} &= x^n + \sum_{k=1}^{n-1} \left(\binom{n}{k} - \binom{n-1}{k} \right) x^k \\ &= x^n + x \sum_{k=1}^{n-1} \binom{n-1}{k-1} x^{k-1} \\ &= x^n + x \sum_{j=0}^{n-2} \binom{n-1}{j} x^j \\ &= x^n + x(S_{n-1} - x^{n-1}) = xS_{n-1}, \end{aligned}$$

that is,

$$S_n = (1+x)S_{n-1}.$$

(Notice the changing of the summation index with $k - 1 = j$; when k runs from 1 to $n-1$, j runs from 0 to $n-2$.) This leads to

$$\sum_{k=0}^{n} \binom{n}{k} x^k = S_n = \prod_{k=1}^{n} \frac{S_k}{S_{k-1}} = \prod_{k=1}^{n} (1+x) = (1+x)^n$$

(see below how to telescope a product; do not forget $S_0 = 1$). Or, if we want to avoid the situation when some S_k is zero, we just use induction based on

the recurrence formula that we found. Induction can also be used if we add all equalities $S_k - S_{k-1} = xS_{k-1}$ for $k = 1, 2, \ldots, n$ in order to get

$$S_n - 1 = \sum_{k=1}^{n}(S_k - S_{k-1}) = x\sum_{k=1}^{n} S_{k-1},$$

hence another recurrence relation for the sums S_n:

$$S_n = 1 + x(S_0 + S_1 + \cdots + S_{n-1}),\ n \geq 1.$$

Note that if we replace $x = \frac{b}{a}$ in the formula $S_n = (1+x)^n$ and then multiply by a^n we find the binomial formula (or theorem) for the expansion of the binomial $a + b$ raised to the nth power:

$$(a+b)^n = \sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^k,$$

which clearly holds for $a = 0$, too, although $a = 0$ is not allowed when considering $x = \frac{b}{a}$. The appearance of the numbers $\binom{n}{k}$ in the binomial formula explains why they are called binomial coefficients.

Again, we used the telescoping method (for a product, or for a sum) in a way that seems not to be very obvious. That is why we tried to illustrate a few more methods for evaluating sums, as induction and the use of simple algebraic rules. We will see other sums (some more general than some of those presented above) and other methods in the following chapters of the book.

Before we go on, we give a few more examples; we have

$$\sum_{k=1}^{n}\frac{1}{k(k+1)} = \sum_{k=1}^{n}\left(\frac{1}{k} - \frac{1}{k+1}\right) = 1 - \frac{1}{n+1} = \frac{n}{n+1},$$

and

$$\begin{aligned}
\sum_{k=1}^{n}\frac{1}{k(k+1)(k+2)} &= \frac{1}{2}\sum_{k=1}^{n}\left(\frac{1}{k(k+1)} - \frac{1}{(k+1)(k+2)}\right) \\
&= \frac{1}{2}\left(\frac{1}{2} - \frac{1}{(n+1)(n+2)}\right) \\
&= \frac{n(n+3)}{4(n+1)(n+2)}.
\end{aligned}$$

And here is our first example of infinite sum:

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)} = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{k(k+1)} = \lim_{n\to\infty} \left(1 - \frac{1}{n+1}\right) = 1.$$

Please verify that

$$\sum_{k=m}^{n} \frac{1}{k(k+1)} = \frac{1}{m} - \frac{1}{n+1} \quad (m \leq n),$$

$$\sum_{k=m}^{\infty} \frac{1}{k(k+1)} = \frac{1}{m},$$

and that

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)(k+2)} = \frac{1}{4}.$$

What is the value of

$$\sum_{k=m}^{\infty} \frac{1}{k(k+1)(k+2)}?$$

Notice also that we used a slightly different (but not *essentially* different) telescoping formula, namely

$$\sum_{k=1}^{n} (b_k - b_{k+1}) = b_1 - b_{n+1}.$$

There are, of course, many possibilities for telescoping. For example, check that

$$\sum_{k=1}^{n} (b_k - b_{k+2}) = b_1 + b_2 - b_{n+1} - b_{n+2}.$$

Finally in this introduction we will see a few products that telescope.

For telescoping a product $\prod_{k=1}^{n} a_k$ we would like to have $a_k = b_{k+1}/b_k$ for every

$k = 1, 2, \ldots, n$, with nonzero numbers $b_1, b_2, \ldots, b_n$, then use the formula that we have already seen

$$\prod_{k=1}^{n} a_k = \prod_{k=1}^{n} \frac{b_{k+1}}{b_k} = \frac{b_{n+1}}{b_1}.$$

Of course, in some situations, we can also use

$$\prod_{k=1}^{n} \frac{b_k}{b_{k+1}} = \frac{b_1}{b_{n+1}},$$

or other similar formulae.
We have, for example,

$$\prod_{k=1}^{n} \left(1 + \frac{1}{k}\right) = \prod_{k=1}^{n} \frac{k+1}{k} = n + 1,$$

where $a_k = 1 + 1/k$ and $b_k = k$ for all k. Also

$$\prod_{k=2}^{n} \left(1 - \frac{1}{k}\right) = \prod_{k=2}^{n} \frac{k-1}{k} = \frac{1}{n}$$

(if we allow $k = 1$, the product is trivially 0, because its first factor is 0), and, consequently,

$$\prod_{k=2}^{\infty} \left(1 - \frac{1}{k}\right) = \lim_{n\to\infty} \prod_{k=2}^{n} \left(1 - \frac{1}{k}\right) = \lim_{n\to\infty} \frac{1}{n} = 0.$$

Further we can find

$$\begin{aligned}\prod_{k=2}^{n} \left(1 - \frac{1}{k^2}\right) &= \prod_{k=2}^{n} \left(\left(1 - \frac{1}{k}\right)\left(1 + \frac{1}{k}\right)\right) \\ &= \prod_{k=2}^{n} \left(1 - \frac{1}{k}\right) \prod_{k=2}^{n} \left(1 + \frac{1}{k}\right) \\ &= \frac{1}{n} \cdot \frac{n+1}{2} = \frac{n+1}{2n},\end{aligned}$$

yielding

$$\prod_{k=2}^{\infty}\left(1-\frac{1}{k^2}\right)=\frac{1}{2}.$$

We invite the reader who is less familiar to the material to do all the computations that we omit as being "evident", or "obvious", or "clear" and so on (and, in general, to do all the computations). Also, we strongly advise the reader to repeat things (which we will also do, from time to time).
A very interesting example for the beginners is

$$\prod_{k=1}^{\infty}\left(1+\frac{1}{a^{2^k}}\right),$$

where a is a real (or even complex) number of absolute value greater than 1 (therefore a is nonzero).
Of course, we look first at the finite case

$$\prod_{k=1}^{n}\left(1+\frac{1}{a^{2^k}}\right).$$

Again, a simple formula is of real help, namely $a^2-b^2=(a-b)(a+b)$ but we use it in a particular case, and in a slightly different form, more precisely we use

$$1+a=\frac{1-a^2}{1-a},\quad a\neq 1.$$

Thus we have

$$\prod_{k=1}^{n}\left(1+\frac{1}{a^{2^k}}\right)=\prod_{k=1}^{n}\frac{1-\dfrac{1}{a^{2^{k+1}}}}{1-\dfrac{1}{a^{2^k}}}=\frac{1-\dfrac{1}{a^{2^{n+1}}}}{1-\dfrac{1}{a^{2^1}}}.$$

Now we see why the condition $|a|>1$ is given: it ensures $|1/a|<1$, hence

$$\lim_{n\to\infty}\left(\frac{1}{a}\right)^{x_n}=0$$

whenever $(x_n)_{n\geq 1}$ is a sequence of real numbers with limit ∞. In particular

$$\prod_{k=1}^{\infty}\left(1+\frac{1}{a^{2^k}}\right)=\lim_{n\to\infty}\frac{1-\dfrac{1}{a^{2^{n+1}}}}{1-\dfrac{1}{a^{2^1}}}=\frac{1}{1-\dfrac{1}{a^2}}=\frac{a^2}{a^2-1}.$$

We end this introductory part with a question related to the problem from which we started. For

$$f(x)=x-\left\lfloor\frac{x}{2}\right\rfloor$$

and n_k defined by $n_1=n$ (an arbitrary positive integer) and $n_k=f(n_{k-1})$ for $k\geq 2$ we have seen that

$$\left\lfloor\frac{n_1}{2}\right\rfloor+\left\lfloor\frac{n_2}{2}\right\rfloor+\cdots=n-1.$$

Can we compute this sum by telescoping? (The answer is yes. So, find how.)

Chapter 1

Telescoping Sums and Products in Algebra

One of the most useful techniques for computing sums and products is the use of the identity

$$(a_2 - a_1) + (a_3 - a_2) + \cdots + (a_{n+1} - a_n) = a_{n+1} - a_1,$$

valid for any complex numbers $a_1, \ldots, a_n$. Therefore, if we need to compute a sum $\sum_{k=1}^{n} b_k$, we might try to find numbers $a_1, \ldots, a_{n+1}$ such that

$$b_1 = a_2 - a_1,\ b_2 = a_3 - a_2, \ldots, b_n = a_{n+1} - a_n$$

and then apply the previous identity to deduce that the sum we are looking for is simply $a_{n+1} - a_1$. If we can do that, we say that the sum is telescopic, or that it telescopes (as shown in the introduction). Finding the numbers $a_1, \ldots, a_n$ is the hard part of the game and lots of practice is certainly helpful! Note that it is always possible to find $a_1, \ldots, a_{n+1}$ as above, namely choose $a_1 = 0$, then $a_2 = b_1$, $a_3 = b_1 + b_2, \ldots$, $a_{n+1} = b_1 + \cdots + b_n$. Of course, this is not very satisfying for our needs...
Let us start with a few classical examples. You certainly know the following

identities

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2},$$
$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6},$$
$$\sum_{k=1}^{n} k^3 = \left[\frac{n(n+1)}{2}\right]^2.$$

What about $\sum_{k=1}^{n} k^4$, or $\sum_{k=1}^{n} k^5$, or more generally $\sum_{k=1}^{n} k^N$ with N a given positive integer? It turns out that one can actually find similar formulae for these sums, but the formulae become fairly complicated when N is large. The key idea is the use of the binomial theorem in the form

$$(k+1)^{N+1} - k^{N+1} = \binom{N+1}{1}k^N + \binom{N+1}{2}k^{N-1} + \cdots + \binom{N+1}{N}k + 1.$$

Thus, adding these relations for $k = 1, 2, \ldots, n$ yields (you see the telescope, don't you?)

$$\begin{aligned}(n+1)^{N+1} - 1 &= \binom{N+1}{1}(1^N + 2^N + \cdots + n^N) \\ &+ \binom{N+1}{2}(1^{N-1} + 2^{N-1} + \cdots + n^{N-1}) + \cdots \\ &+ \binom{N+1}{N}(1 + 2 + \cdots + n) + n.\end{aligned}$$

This shows that if we can compute $1^j + 2^j + \cdots + n^j$ for $j = 1, 2, \ldots, N-1$, then we can also compute it for $j = N$. For instance, take $N = 1$, then the previous identity becomes

$$(n+1)^2 - 1 = 2(1 + 2 + \cdots + n) + n,$$

and a simple algebraic manipulation shows that this recovers the classical formula

$$1+2+\cdots+n=\frac{n(n+1)}{2}.$$

Next, take $N=2$, then we obtain

$$(n+1)^3-1=3(1^2+\cdots+n^2)+3(1+2+\cdots+n)+n,$$

in other words

$$\begin{aligned}
1^2+\cdots+n^2&=\frac{(n+1)^3-1-3(1+2+\cdots+n)-n}{3}\\
&=\frac{n^3+3n^2+3n-\dfrac{3n(n+1)}{2}-n}{3}\\
&=\frac{2n^3+6n^2+6n-3n^2-3n-2n}{6}\\
&=\frac{2n^3+3n^2+n}{6}=\frac{n(2n^2+3n+1)}{6}\\
&=\frac{n(n+1)(2n+1)}{6}.
\end{aligned}$$

Example 1.1. *Prove that*

$$\sum_{k=1}^{n}k^4=\frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}.$$

Solution. From the binomial theorem we have

$$(k+1)^5-k^5=5k^4+10k^3+10k^2+5k+1,$$

thus it follows that

$$5\sum_{k=1}^{n}k^4+10\sum_{k=1}^{n}k^3+10\sum_{k=1}^{n}k^2+5\sum_{k=1}^{n}k+\sum_{k=1}^{n}1=\sum_{k=1}^{n}((k+1)^5-k^5)=(n+1)^5-1.$$

Hence

$$5\sum_{k=1}^{n}k^4=(n+1)^5-10\cdot\frac{n^2(n+1)^2}{4}-10\cdot\frac{n(n+1)(2n+1)}{6}-5\cdot\frac{n(n+1)}{2}-(n+1),$$

yielding

$$\sum_{k=1}^{n} k^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}.$$

Example 1.2. (IMO Longlist 1977) *Evaluate*

$$\sum_{k=1}^{n} k(k+1)\cdots(k+p-1),$$

where n and p are positive integers.

Solution. We have

$$\begin{aligned}\sum_{k=1}^{n} k(k+1)\cdots(k+p-1) &= \sum_{k=1}^{n} \frac{k(k+1)\cdots(k+p)-(k-1)k\cdots(k+p-1)}{p+1} \\ &= \frac{n(n+1)\cdots(n+p)}{p+1}.\end{aligned}$$

For example

$$\sum_{k=1}^{n} k(k+1) = \frac{n(n+1)(n+2)}{3}$$

(as we already have seen in the introduction) and

$$\sum_{k=1}^{n} k(k+1)(k+2) = \frac{n(n+1)(n+2)(n+3)}{4}.$$

Example 1.3. *For positive integers n and $p \geq 2$ evaluate*

$$\sum_{k=1}^{n} \frac{1}{k(k+1)\cdots(k+p-1)}.$$

Solution. We have

$$\sum_{k=1}^{n} \frac{1}{k(k+1)\cdots(k+p-1)} = \frac{1}{p-1}\sum_{k=1}^{n} \frac{(k+p-1)-k}{k(k+1)\cdots(k+p-1)}$$

$$= \frac{1}{p-1}\sum_{k=1}^{n}\left(\frac{1}{k(k+1)\cdots(k+p-2)} - \frac{1}{(k+1)\cdots(k+p-1)}\right)$$

$$= \frac{1}{p-1}\left(\frac{1}{(p-1)!} - \frac{1}{(n+1)\cdots(n+p-1)}\right).$$

For instance,

$$\sum_{k=1}^{n}\frac{1}{k(k+1)} = 1 - \frac{1}{n+1}$$

and

$$\sum_{k=1}^{n}\frac{1}{k(k+1)(k+2)} = \frac{1}{2}\left(\frac{1}{2} - \frac{1}{(n+1)(n+2)}\right).$$

Example 1.4. (IMO Longlist 1970) *For even positive integer n, prove that*

$$\sum_{i=1}^{n}(-1)^{i+1}\cdot\frac{1}{i} = 2\sum_{i=1}^{n/2}\frac{1}{n+2i}.$$

Solution. Induction on n works. It is easy to verify the case $n=2$, so assume the result is true for some even natural number n. Then

$$\sum_{i=1}^{n+2}(-1)^{i+1}\cdot\frac{1}{i} = \frac{1}{n+1} - \frac{1}{n+2} + \sum_{i=1}^{n}(-1)^{i+1}\cdot\frac{1}{i}$$

and by the inductive assumption, this will be equal to

$$\frac{1}{n+1} - \frac{1}{n+2} + \frac{2}{n+2} + 2\left(\frac{1}{n+4} + \cdots + \frac{1}{n+n}\right).$$

Since

$$\frac{1}{n+1} + \frac{1}{n+2} = 2\left(\frac{1}{(n+2)+n} + \frac{1}{(n+2)+(n+2)}\right)$$

the sum becomes $2\sum_{i=1}^{\frac{n+2}{2}}\frac{1}{n+2i}$ which finishes the proof.

The next example involves factorials.

Example 1.5. *Evaluate*

$$1^2 \cdot 2! + 2^2 \cdot 3! + \cdots + n^2(n+1)!$$

Solution. We try to find numbers a_k such that

$$k^2(k+1)! = a_{k+1} - a_k \text{ for } 1 \leq k \leq n.$$

It is natural to look for a_k of the form $a_k = k!b_k$ for some number b_k. Indeed, the previous relation becomes then the much easier

$$k^2(k+1) = (k+1)b_{k+1} - b_k.$$

Next, we try to find b_k of the form $b_k = P(k)$ for some polynomial P. Thus we ask that

$$k^2(k+1) = (k+1)P(k+1) - P(k).$$

If we are lucky, the previous equality holds for all integers k and so, by considering degrees P must be quadratic and monic, say

$$P(X) = X^2 + cX + d.$$

Plugging in the previous relation yields

$$k^2(k+1) = (k+1)((k+1)^2 + c(k+1) + d) - k^2 - ck - d.$$

Identifying coefficients easily yields $c = -1$ and $d = -2$. Thus

$$\begin{aligned}\sum_{k=1}^{n} k^2(k+1)! &= \sum_{k=1}^{n}[(k+1)!((k+1)^2 - (k+1) - 2) - k!(k^2 - k - 2)] \\ &= (n+2)(n-1)(n+1)! - (1+1)(1-2)1! \\ &= (n-1)(n+2)! + 2.\end{aligned}$$

This example illustrates an extremely important technique for finding telescoping sums and products. We did not immediately see the correct formula, so instead we guessed what the form of the answer might look like. Since we weren't sure of the exact form, we guessed a form with some undetermined parameters (in this case, the degree of P and its coefficients). We then used the fact that we wanted the sum to telescope to let us solve for these parameters. Since we guessed the correct form, we were successful and we solved the problem.

Example 1.6. *Evaluate*

$$\sum_{k=1}^{n} k!(k^2+k+1).$$

Solution. We try to write

$$k!(k^2+k+1) = a_{k+1} - a_k$$

for some numbers a_k. To get rid of $k!$, let us choose $a_k = k!b_k$ for some numbers b_k that we still have to find. Then

$$a_{k+1} - a_k = (k+1)!b_{k+1} - k!b_k = k!((k+1)b_{k+1} - b_k),$$

so we need

$$(k+1)b_{k+1} - b_k = k^2+k+1.$$

But there is an obvious choice: set $b_k = k$ for all k. Thus we can take $a_k = k \cdot k!$ and the desired sum telescopes nicely:

$$\sum_{k=1}^{n} k!(k^2+k+1) = \sum_{k=1}^{n} [(k+1)!(k+1) - k!k] = (n+1)!(n+1) - 1.$$

Example 1.7. *Evaluate*

$$\sum_{k=2}^{n} \frac{3k^2-1}{(k^3-k)^2}.$$

Solution. First, observe that

$$\frac{3k^2-1}{(k^3-k)^2} = \frac{k-\frac{1}{2}}{k^2(k-1)^2} - \frac{k+\frac{1}{2}}{k^2(k+1)^2}.$$

Now the series telescopes as

$$\sum_{k=2}^{n} \frac{3k^2-1}{(k^3-k)^2} = \frac{3}{8} - \frac{n+\frac{1}{2}}{n^2(n+1)^2}.$$

Example 1.8. *Compute*

$$\frac{1\cdot 3!}{3}+\frac{2\cdot 4!}{3^2}+\cdots+\frac{n\cdot(n+2)!}{3^n}.$$

Solution. Observe that

$$\frac{k(k+2)!}{3^k}=\frac{(k+3-3)(k+2)!}{3^k}=\frac{(k+3)(k+2)!-3(k+2)!}{3^k}$$
$$=\frac{(k+3)!-3(k+2)!}{3^k}=\frac{(k+3)!}{3^k}-\frac{(k+2)!}{3^{k-1}}.$$

We obtain therefore a telescoping sum

$$\sum_{k=1}^{n}\frac{k(k+2)!}{3^k}=\sum_{k=1}^{n}\left(\frac{(k+3)!}{3^k}-\frac{(k+2)!}{3^{k-1}}\right)=\frac{(n+3)!}{3^n}-\frac{3!}{3^0}=\frac{(n+3)!}{3^n}-6.$$

Example 1.9. *Evaluate*

$$\sum_{k=1}^{n}\frac{4k+\sqrt{4k^2-1}}{\sqrt{2k+1}+\sqrt{2k-1}}.$$

Solution. This problem requires some algebraic skills. To simplify the formulae let us denote

$$a=\sqrt{2k+1},\quad b=\sqrt{2k-1}.$$

Then $2k+1=a^2$ and $2k-1=b^2$, so that $4k=a^2+b^2$ and $4k^2-1=a^2b^2$. Thus

$$\frac{4k+\sqrt{4k^2-1}}{\sqrt{2k+1}+\sqrt{2k-1}}=\frac{a^2+b^2+ab}{a+b}=\frac{a^2+ab+b^2}{a+b}=\frac{\dfrac{a^3-b^3}{a-b}}{a+b}=\frac{a^3-b^3}{a^2-b^2}.$$

Since

$$a^2-b^2=(2k+1)-(2k-1)=2,$$

we obtain

$$\frac{4k+\sqrt{4k^2-1}}{\sqrt{2k+1}+\sqrt{2k-1}}=\frac{a^3-b^3}{2}=\frac{\sqrt{(2k+1)^3}-\sqrt{(2k-1)^3}}{2}.$$

Telescoping, we find

$$\sum_{k=1}^{n} \frac{4k+\sqrt{4k^2-1}}{\sqrt{2k+1}+\sqrt{2k-1}} = \frac{\sqrt{(2n+1)^3}-1}{2}.$$

Example 1.10. *Evaluate the sum*

$$\frac{2}{3+1} + \frac{2^2}{3^2+1} + \frac{2^3}{3^4+1} + \cdots + \frac{2^{n+1}}{3^{2^n}+1}.$$

Solution. We know that

$$\frac{1}{a+1} = \frac{a-1}{a^2-1} = -\frac{1}{a^2-1} + \frac{1}{2}\left(\frac{1}{a+1} + \frac{1}{a-1}\right),$$

which implies

$$\frac{1}{2}\cdot\frac{1}{a+1} = \frac{1}{2(a-1)} - \frac{1}{a^2-1}.$$

Applying this identity with $a = 3^{2^k}$, we obtain

$$\frac{1}{2(3^{2^k}+1)} = \frac{1}{2(3^{2^k}-1)} - \frac{1}{3^{2^{k+1}}-1}.$$

Multiplying this by 2^{k+2}, we obtain the relation

$$\frac{2^{k+1}}{3^{2^k}+1} = \frac{2^{k+1}}{3^{2^k}-1} - \frac{2^{k+2}}{3^{2^{k+1}}-1}.$$

Thus

$$\begin{aligned}\frac{2}{3+1} + \frac{2^2}{3^2+1} + \frac{2^3}{3^4+1} + \cdots + \frac{2^{n+1}}{3^{2^n}+1} &= \sum_{k=0}^{n}\left(\frac{2^{k+1}}{3^{2^k}-1} - \frac{2^{k+2}}{3^{2^{k+1}}-1}\right)\\ &= 1 - \frac{2^{n+2}}{3^{2^{n+1}}-1},\end{aligned}$$

and we are done.

Example 1.11. *Evaluate*

$$\sum_{k=1}^{n} \frac{4k}{4k^4+1}.$$

Solution. The key ingredient in solving this problem is realizing that the denominator factors rather nicely. Indeed,

$$\begin{aligned}4k^4+1 &= 4k^4+4k^2+1-4k^2 = (2k^2+1)^2-(2k)^2\\ &= (2k^2-2k+1)(2k^2+2k+1).\end{aligned}$$

On the other hand, we observe that the numerator $4k$ is simply the difference between $2k^2+2k+1$ and $2k^2-2k+1$. Therefore

$$\begin{aligned}\sum_{k=1}^{n} \frac{4k}{4k^4+1} &= \sum_{k=1}^{n} \frac{(2k^2+2k+1)-(2k^2-2k+1)}{(2k^2+2k+1)(2k^2-2k+1)}\\ &= \sum_{k=1}^{n} \left(\frac{1}{2k^2-2k+1} - \frac{1}{2k^2+2k+1}\right).\end{aligned}$$

We almost have a telescopic sum: letting $a_k = 2k^2-2k+1$, the only extra observation we need is that

$$2k^2+2k+1 = a_{k+1}.$$

Indeed, $a_k = 2k(k-1)+1$ so

$$a_{k+1} = 2(k+1)k+1 = 2k^2+2k+1,$$

as desired. Therefore we have a telescopic sum

$$\begin{aligned}\sum_{k=1}^{n} \frac{4k}{4k^4+1} &= \sum_{k=1}^{n} \left(\frac{1}{2k^2-2k+1} - \frac{1}{2(k+1)^2-2(k+1)+1}\right)\\ &= 1 - \frac{1}{2n^2+2n+1} = \frac{2n^2+2n}{2n^2+2n+1},\end{aligned}$$

and we are done.

Example 1.12. *Let $a_1, a_2, \ldots, a_n$ be positive real numbers such that*

$$a_1 a_2 \cdots a_n = 1.$$

Prove that

$$\frac{a_1}{1+a_1} + \frac{a_2}{(1+a_1)(1+a_2)} + \frac{a_3}{(1+a_1)(1+a_2)(1+a_3)} + \cdots + \frac{a_n}{(1+a_1)(1+a_2)\cdots(1+a_n)} \ge \frac{2^n - 1}{2^n}.$$

Solution. The sum in the left hand-side is actually fairly easy to compute once we realize that

$$\frac{a_k}{(1+a_1)\cdots(1+a_k)} = \frac{1+a_k-1}{(1+a_1)\cdots(1+a_k)} = \frac{1}{(1+a_1)\cdots(1+a_{k-1})} - \frac{1}{(1+a_1)\cdots(1+a_k)}.$$

Therefore the sum is telescopic and

$$\sum_{k=1}^{n} \frac{a_k}{(1+a_1)\cdots(1+a_k)} = 1 - \frac{1}{1+a_1} + \frac{1}{1+a_1} - \frac{1}{(1+a_1)(1+a_2)} + \cdots + \frac{1}{(1+a_1)\cdots(1+a_{n-1})} - \frac{1}{(1+a_1)\cdots(1+a_n)} = 1 - \frac{1}{(1+a_1)\cdots(1+a_n)}.$$

Since

$$\frac{2^n-1}{2^n} = 1 - \frac{1}{2^n},$$

the problem reduces in the end to the inequality

$$\frac{1}{2^n} \ge \frac{1}{(1+a_1)\cdots(1+a_n)},$$

which is equivalent to

$$(1+a_1)\cdots(1+a_n) \ge 2^n.$$

This is a fairly standard consequence of the AM-GM inequality, since

$$1+a_1 \geq 2\sqrt{a_1},\ 1+a_2 \geq 2\sqrt{a_2}, \ldots, 1+a_n \geq 2\sqrt{a_n},$$

thus

$$(1+a_1)\cdots(1+a_n) \geq 2^n\sqrt{a_1\cdots a_n} = 2^n.$$

Example 1.13. *Evaluate*

$$\frac{2^1}{4^1-1}+\frac{2^2}{4^2-1}+\frac{2^4}{4^4-1}+\frac{2^8}{4^8-1}+\cdots$$

Solution. We are asked to evaluate

$$\sum_{n=0}^{\infty}\frac{2^{2^n}}{4^{2^n}-1} = \sum_{n=0}^{\infty}\frac{2^{2^n}}{2^{2^{n+1}}-1}.$$

First we have

$$\begin{aligned}\frac{2^{2^n}}{2^{2^{n+1}}-1} &= \frac{2^{2^n}}{(2^{2^n}+1)(2^{2^n}-1)}\\ &= \frac{2^{2^n}+1-1}{(2^{2^n}+1)(2^{2^n}-1)}\\ &= \frac{1}{2^{2^n}-1}-\frac{1}{2^{2^{n+1}}-1},\end{aligned}$$

which implies

$$\begin{aligned}\sum_{n=0}^{N}\frac{2^{2^n}}{2^{2^{n+1}}-1} &= \sum_{n=0}^{N}\left(\frac{1}{2^{2^n}-1}-\frac{1}{2^{2^{n+1}}-1}\right)\\ &= \frac{1}{2^{2^0}-1}-\frac{1}{2^{2^{N+1}}-1} = 1-\frac{1}{2^{2^{N+1}}-1}.\end{aligned}$$

Thus,

$$\sum_{n=0}^{\infty}\frac{2^{2^n}}{2^{2^{n+1}}-1} = \lim_{N\to\infty}\sum_{n=0}^{N}\frac{2^{2^n}}{2^{2^{n+1}}-1} = \lim_{N\to\infty}\left(1-\frac{1}{2^{2^{N+1}}-1}\right) = 1.$$

Example 1.14. (IMC 2015) *Define a sequence* $(F(n))_{n\geq 0}$ *by*

$$F(0)=0,\ F(1)=\frac{3}{2},\ \text{and } F(n)=\frac{5}{2}F(n-1)-F(n-2)$$

for $n \geq 2$. *Is* $\displaystyle\sum_{n=0}^{\infty}\frac{1}{F(2^n)}$ *a rational number?*

Solution. Let us start by finding the general term of the sequence $(F(n))_{n\geq 0}$. Note that the recurrence relation can be written

$$F(n)=2F(n-1)+\frac{F(n-1)}{2}-F(n-2),$$

in other words

$$F(n)-2F(n-1)=\frac{F(n-1)-2F(n-2)}{2}.$$

The sequence $G(n):=F(n)-2F(n-1)$ therefore satisfies

$$G(n)=\frac{G(n-1)}{2}$$

and by an immediate induction (or using a telescopic product) we obtain

$$G(n)=\frac{G(1)}{2^{n-1}}=\frac{3}{2^n}.$$

Thus

$$F(n)-2F(n-1)=\frac{3}{2^n}$$

and dividing by 2^n yields

$$\frac{F(n)}{2^n}-\frac{F(n-1)}{2^{n-1}}=\frac{3}{4^n}.$$

Adding up these relations for $n=1,2,\ldots,N$ and recognizing a telescopic sum on the left-hand side yields

$$\frac{F(N)}{2^N}=\sum_{n=1}^{N}\frac{3}{4^n}=\sum_{n=1}^{N}\left(\frac{1}{4^{n-1}}-\frac{1}{4^n}\right)=1-\frac{1}{4^N},$$

which yields

$$F(N) = 2^N - 2^{-N}.$$

It follows that

$$\frac{1}{F(2^n)} = \frac{1}{2^{2^n} - 2^{-2^n}} = \frac{2^{2^n}}{2^{2^{n+1}} - 1} = \frac{1}{2^{2^n} - 1} - \frac{1}{2^{2^{n+1}} - 1}.$$

We recognize the telescopic sum from the previous example and we conclude that

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{1}{F(2^n)} &= \lim_{N\to\infty} \sum_{n=0}^{N} \frac{1}{F(2^n)} \\ &= \lim_{N\to\infty} \sum_{n=0}^{N} \left(\frac{1}{2^{2^n} - 1} - \frac{1}{2^{2^{n+1}} - 1} \right) \\ &= \lim_{N\to\infty} \left(1 - \frac{1}{2^{2^{N+1}} - 1} \right) = 1,\end{aligned}$$

which is a rational number.

Example 1.15. *Let*

$$h_n = 1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2n-1} \text{ for } n \in \mathbb{N}^*.$$

Prove that:

$$\frac{1}{h_1^2} + \frac{1}{3h_2^2} + \cdots + \frac{1}{(2n-1)h_n^2} < 2$$

for any positive integer n.

Solution. We have

$$h_k - h_{k-1} = \frac{1}{2k-1} \quad \text{and} \quad h_{k-1} < h_k,$$

therefore

$$\frac{1}{(2k-1)h_k^2} = \frac{h_k - h_{k-1}}{h_k^2} < \frac{h_k - h_{k-1}}{h_k h_{k-1}} = \frac{1}{h_{k-1}} - \frac{1}{h_k}$$

for $n \geq 2$. Now sum these inequalities in order to get

$$\sum_{k=1}^{n} \frac{1}{(2k-1)h_k^2} = 1 + \sum_{k=2}^{n} \frac{1}{(2k-1)h_k^2} < 1 + \sum_{k=2}^{n} \left(\frac{1}{h_{k-1}} - \frac{1}{h_k} \right)$$
$$= 1 + \frac{1}{h_1} - \frac{1}{h_n} = 2 - \frac{1}{h_n} < 2,$$

as required.

Example 1.16. *Find and prove a simple formula for the sum*

$$S = \frac{1^3}{1^4+4} - \frac{3^3}{3^4+4} + \frac{5^3}{5^4+4} - \cdots + \frac{(-1)^n \cdot (2n+1)^3}{(2n+1)^4+4}.$$

Solution. We have

$$a^4 + 4b^4 = a^4 + 4a^2b^2 + 4b^4 - 4a^2b^2 = (a^2+2b^2)^2 - (2ab)^2$$
$$= (a^2 - 2ab + 2b^2)(a^2 + 2ab + 2b^2)$$

a formula that we will often meet, in various appearances. In this case we use it for

$$(2k+1)^4 + 4 = ((2k+1)^2 - 2(2k+1) + 2)((2k+1)^2 + 2(2k+1) + 2)$$
$$= (4k^2+1)(4k^2+8k+5).$$

Because $4k^2 + 8k + 5 = 4(k+1)^2 + 1$ we can hope that the decomposition as a sum of simple fractions of the general term of the sum (without the sign),

$$\frac{(2k+1)^3}{(2k+1)^4+1} = \frac{(2k+1)^3}{(4k^2+1)(4k^2+8k+5)}$$

will give something useful. And indeed, if we write

$$\frac{(2k+1)^3}{(4k^2+1)(4k^2+8k+5)} = \frac{Ak+B}{4k^2+1} + \frac{Ck+D}{4k^2+8k+5}$$

and identify coefficients we find that

$$\begin{aligned}\frac{(2k+1)^3}{(4k^2+1)(4k^2+8k+5)} &= \frac{k}{4k^2+1} + \frac{k+1}{4k^2+8k+5} \\ &= \frac{k}{4k^2+1} + \frac{k+1}{4(k+1)^2+1}\end{aligned}$$

which immediately permits the evaluation of the sum by telescoping:

$$\begin{aligned}S &= \sum_{k=0}^{n}(-1)^k\left(\frac{k}{4k^2+1} + \frac{k+1}{4(k+1)^2+1}\right) \\ &= \sum_{k=0}^{n}\left((-1)^k\frac{k}{4k^2+1} - (-1)^{k+1}\frac{k+1}{4(k+1)^2+1}\right) \\ &= -(-1)^{n+1}\cdot\frac{n+1}{4(n+1)^2+1} = (-1)^n\cdot\frac{n+1}{4n^2+8n+5}.\end{aligned}$$

Example 1.17. *Calculate the sum*

$$\sum_{k=1}^{n}\frac{1}{\sqrt{k+\sqrt{k^2-1}}}.$$

Solution. We need to understand the expression $\dfrac{1}{\sqrt{k+\sqrt{k^2-1}}}$. For this, let us denote $a = k + \sqrt{k^2-1}$ and consider its conjugate expression $b = k - \sqrt{k^2-1}$. Then $a + b = 2k$ and

$$ab = k^2 - \sqrt{k^2-1}^{\,2} = k^2 - (k^2-1) = 1.$$

Thus

$$\frac{1}{\sqrt{k+\sqrt{k^2-1}}} = \frac{1}{\sqrt{a}} = \sqrt{b}.$$

On the other hand, let us observe that

$$\begin{aligned}
b = k - \sqrt{k^2 - 1} &= k - \sqrt{(k+1)(k-1)} \\
&= \frac{k+1+k-1}{2} - \sqrt{(k+1)(k-1)} \\
&= \frac{k+1-2\sqrt{(k+1)(k-1)}+k-1}{2} \\
&= \frac{\left(\sqrt{k+1} - \sqrt{k-1}\right)^2}{2},
\end{aligned}$$

hence

$$\sqrt{b} = \frac{\sqrt{k+1} - \sqrt{k-1}}{\sqrt{2}} = \frac{\sqrt{k+1} - \sqrt{k} + \sqrt{k} - \sqrt{k-1}}{\sqrt{2}}.$$

We recognize two telescopic sums and combining the previous observations we obtain

$$\begin{aligned}
\sum_{k=1}^{n} \frac{1}{\sqrt{k + \sqrt{k^2-1}}} &= \sum_{k=1}^{n} \frac{\sqrt{k+1} - \sqrt{k} + \sqrt{k} - \sqrt{k-1}}{\sqrt{2}} \\
&= \frac{\sqrt{n+1} - 1}{\sqrt{2}} + \frac{\sqrt{n} - \sqrt{0}}{\sqrt{2}} \\
&= \frac{\sqrt{n+1} + \sqrt{n} - 1}{\sqrt{2}}.
\end{aligned}$$

Example 1.18. *Compute the sum*

$$\frac{1}{1^2 \cdot 3^2} + \frac{2}{3^2 \cdot 5^2} + \cdots + \frac{n}{(2n-1)^2 \cdot (2n+1)^2}.$$

Solution. Observe that

$$(2k+1)^2 - (2k-1)^2 = 4k^2 + 4k + 1 - (4k^2 - 4k + 1) = 8k,$$

thus

$$\frac{k}{(2k+1)^2\cdot(2k-1)^2} = \frac{1}{8}\cdot\frac{8k}{(2k+1)^2\cdot(2k-1)^2}$$
$$= \frac{1}{8}\cdot\frac{(2k+1)^2-(2k-1)^2}{(2k+1)^2\cdot(2k-1)^2}$$
$$= \frac{1}{8}\left(\frac{1}{(2k-1)^2}-\frac{1}{(2k+1)^2}\right).$$

We obtain this way a telescopic sum, whose value is

$$\sum_{k=1}^{n}\frac{k}{(2k+1)^2\cdot(2k-1)^2} = \frac{1}{8}\left(\frac{1}{1^2}-\frac{1}{(2n+1)^2}\right) = \frac{n^2+n}{2(2n+1)^2}.$$

Example 1.19. (USAMTS 2002) *Compute*

$$\frac{1}{1\sqrt{2}+2\sqrt{1}}+\frac{1}{2\sqrt{3}+3\sqrt{2}}+\frac{1}{3\sqrt{4}+4\sqrt{3}}+\cdots$$
$$+\frac{1}{4012008\sqrt{4012009}+4012009\sqrt{4012008}}.$$

Solution. We are asked to compute

$$\sum_{k=1}^{4012008}\frac{1}{k\sqrt{k+1}+(k+1)\sqrt{k}}.$$

Let us deal with the general term:

$$\frac{1}{k\sqrt{k+1}+(k+1)\sqrt{k}} = \frac{1}{\sqrt{k}\cdot\sqrt{k+1}\cdot\left(\sqrt{k}+\sqrt{k+1}\right)}$$
$$= \frac{\sqrt{k+1}-\sqrt{k}}{\sqrt{k}\cdot\sqrt{k+1}},$$

where the last equality follows from the simple identity

$$\left(\sqrt{k}+\sqrt{k+1}\right)\left(\sqrt{k+1}-\sqrt{k}\right)=1.$$

We deduce that

$$\frac{1}{k\sqrt{k+1}+(k+1)\sqrt{k}}=\frac{1}{\sqrt{k}}-\frac{1}{\sqrt{k+1}},$$

which shows that our sum is a telescopic one, with value

$$\sum_{k=1}^{4012008}\frac{1}{k\sqrt{k+1}+(k+1)\sqrt{k}}=1-\frac{1}{\sqrt{4012009}}.$$

Finally, note that

$$4012009=4\cdot 10^6+12\cdot 10^3+3^2=(2\cdot 10^3+3)^2=2003^2,$$

thus

$$\frac{1}{1\sqrt{2}+2\sqrt{1}}+\frac{1}{2\sqrt{3}+3\sqrt{2}}+\frac{1}{3\sqrt{4}+4\sqrt{3}}+\cdots$$
$$+\frac{1}{4012008\sqrt{4012009}+4012009\sqrt{4012008}}=1-\frac{1}{2003}=\frac{2002}{2003}.$$

Example 1.20. *Evaluate*

$$\sum_{k=1}^{n}\frac{k^2+k-1}{(k+2)!}.$$

Solution. Let us try to find numbers b_k such that

$$\frac{k^2+k-1}{(k+2)!}=\frac{b_{k-1}}{(k+1)!}-\frac{b_k}{(k+2)!}.$$

Clearing denominators, the previous equality is equivalent to

$$k^2+k-1=(k+2)b_{k-1}-b_k.$$

Let us look for $b_k=mk+n$ with m,n real numbers. We would like to have

$$k^2+k-1=(k+2)(mk+n-m)-(mk+n)$$

for all k. Identifying coefficients yields $m=1$ and $n=1$, in other words $b_k=k+1$. We deduce that

$$\sum_{k=1}^{n}\frac{k^2+k-1}{(k+2)!}=\sum_{k=1}^{n}\left(\frac{k}{(k+1)!}-\frac{k+1}{(k+2)!}\right)=\frac{1}{2}-\frac{n+1}{(n+2)!}.$$

Example 1.21. *Define a sequence* $(x_n)_{n\geq 1}$ *by* $x_1 = \frac{1}{2}$, *and for* $n \geq 1$,

$$x_{n+1} = \frac{x_n + x_n^2}{1 + x_n + x_n^2}.$$

Compute

$$\frac{1}{x_1+1} + \frac{1}{x_2+1} + \frac{1}{x_3+1} + \cdots + \frac{1}{x_{2015}+1} + \frac{1}{x_{2016}}.$$

Solution. The recurrence relation can be written

$$\frac{1}{x_{n+1}} = \frac{1 + x_n + x_n^2}{x_n + x_n^2} = 1 + \frac{1}{x_n(x_n+1)} = 1 + \frac{1}{x_n} - \frac{1}{x_n+1},$$

which can be rearranged as

$$\frac{1}{x_n+1} = 1 + \frac{1}{x_n} - \frac{1}{x_{n+1}}.$$

We deduce that

$$\begin{aligned}\sum_{k=1}^{2015} \frac{1}{x_k+1} &= 2015 + \sum_{k=1}^{2015} \left(\frac{1}{x_k} - \frac{1}{x_{k+1}}\right)\\ &= 2015 + 2 - \frac{1}{x_{2016}}.\end{aligned}$$

This shows that the sum we are asked to compute equals $2015 + 2 = 2017$. Of course, the number 2015 plays no special role in this problem. In the exact same way we can prove that

$$\sum_{k=1}^{n} \frac{1}{x_k+1} + \frac{1}{x_{n+1}} = n + \frac{1}{x_1}$$

for any positive integer n.

Example 1.22. *Prove that for all* $n \geq 1$

$$\sum_{k=1}^{n} \frac{1}{(k+1)\sqrt{k}} < 2.$$

Solution. We have

$$\begin{aligned}\sum_{k=1}^{n}\frac{1}{(k+1)\sqrt{k}} &= \sum_{k=1}^{n}\frac{\sqrt{k}}{k(k+1)} = \sum_{k=1}^{n}\left(\frac{\sqrt{k}}{k}-\frac{\sqrt{k}}{k+1}\right)\\ &= \sum_{k=1}^{n}\frac{\sqrt{k}}{k}-\sum_{k=1}^{n}\frac{\sqrt{k}}{k+1}\\ &= 1+\sum_{k=2}^{n}\frac{\sqrt{k}}{k}-\sum_{k=2}^{n+1}\frac{\sqrt{k-1}}{k}\\ &= 1-\frac{\sqrt{n}}{n+1}+\sum_{k=2}^{n}\frac{\sqrt{k}-\sqrt{k-1}}{k}.\end{aligned}$$

It suffices therefore to prove that

$$\sum_{k=2}^{n}\frac{\sqrt{k}-\sqrt{k-1}}{k}<1.$$

But

$$\frac{\sqrt{k}-\sqrt{k-1}}{k}<\frac{\sqrt{k}-\sqrt{k-1}}{\sqrt{k(k-1)}}=\frac{1}{\sqrt{k-1}}-\frac{1}{\sqrt{k}},$$

thus

$$\sum_{k=2}^{n}\frac{\sqrt{k}-\sqrt{k-1}}{k}<\sum_{k=2}^{n}\left(\frac{1}{\sqrt{k-1}}-\frac{1}{\sqrt{k}}\right)=1-\frac{1}{\sqrt{n}}<1,$$

as needed.

Example 1.23. (IMO Longlist 1970) *Let n be a positive integer. Prove that*

$$1+\frac{1}{2^3}+\frac{1}{3^3}+\cdots+\frac{1}{n^3}<\frac{5}{4}.$$

Solution 1. We have

$$0<(k-1)k(k+1)=k(k^2-1)<k^3$$

for all $k \geq 2$, hence

$$\begin{aligned}
\sum_{k=2}^{n} \frac{1}{k^3} &< \sum_{k=2}^{n} \frac{1}{(k-1)k(k+1)} \\
&= \frac{1}{2} \sum_{k=2}^{n} \left(\frac{1}{(k-1)k} - \frac{1}{k(k+1)} \right) \\
&= \frac{1}{2} \left(\frac{1}{2} - \frac{1}{n(n+1)} \right) < \frac{1}{4},
\end{aligned}$$

which is clearly equivalent to the required inequality.

Solution 2. We prove the stronger inequality

$$1 + \frac{1}{2^3} + \frac{1}{3^3} + \cdots + \frac{1}{n^3} + \frac{1}{4n} \leq \frac{5}{4}$$

for all positive integers $n \geq 1$ (the equal sign only appears when $n \in \{1, 2\}$). This is a common trick in induction proofs of inequalities. A direct induction proof of the desired inequality would fail. So instead one looks for an improved inequality where induction does work. The induction step follows from

$$1 + \frac{1}{2^3} + \frac{1}{3^3} + \cdots + \frac{1}{k^3} + \frac{1}{(k+1)^3} + \frac{1}{4(k+1)} \leq 1 + \frac{1}{2^3} + \frac{1}{3^3} + \cdots + \frac{1}{k^3} + \frac{1}{4k}$$

which is equivalent to

$$\frac{1}{(k+1)^3} \leq \frac{1}{4k} - \frac{1}{4(k+1)} = \frac{1}{4k(k+1)},$$

and to

$$4k \leq (k+1)^2 \Leftrightarrow (k-1)^2 \geq 0.$$

Since this inequality is strict for $k \geq 2$, we will also get

$$1 + \frac{1}{2^3} + \frac{1}{3^3} + \cdots + \frac{1}{n^3} + \frac{1}{4n} < \frac{5}{4}$$

for $n \geq 3$. Anyway, the original inequality

$$1 + \frac{1}{2^3} + \frac{1}{3^3} + \cdots + \frac{1}{n^3} < \frac{5}{4}$$

is definitely strict for all $n \geq 1$.

Example 1.24. (IMO Longlist 1970) *Let $1 < n \in \mathbb{N}$ and $1 \leq a \in \mathbb{R}$ and suppose there are n positive numbers x_i, $i \in \mathbb{N}$, $1 \leq i \leq n$ such that $x_1 = 1$ and $x_i/x_{i-1} = a + \alpha_i$ for $2 \leq i \leq n$, where $\alpha_i \leq 1/i(i+1)$. Prove that*

$$\sqrt[n-1]{x_n} < a + \frac{1}{n-1}.$$

Solution. Note that

$$x_n = x_{n-1}(a + \alpha_n) = \cdots = (a + \alpha_2)(a + \alpha_3)\cdots(a + \alpha_n).$$

Therefore we obtain

$$\begin{aligned}(n-1)\sqrt[n-1]{x_n} &= (n-1)\sqrt[n-1]{(a+\alpha_2)(a+\alpha_3)\cdots(a+\alpha_n)} \\ &\overset{AM-GM}{\leq} (n-1)a + \sum_{i=2}^{n} \alpha_i.\end{aligned}$$

Therefore it is sufficient to check that

$$\alpha_2 + \alpha_3 + \cdots + \alpha_n < 1.$$

Noting that

$$\alpha_i \leq \frac{1}{i(i+1)} = \frac{1}{i} - \frac{1}{i+1},$$

we obtain

$$\sum_{i=2}^{n} \alpha_i \leq \sum_{i=2}^{n} \left(\frac{1}{i} - \frac{1}{i+1}\right) = \frac{1}{2} - \frac{1}{n+1} < 1.$$

Equality does not hold since $n \neq -3$, and all the α_i cannot be equal.

Example 1.25. *Let $n > 1$ and defined $a_0 = 1/2$ and, for $0 \leq k \leq n-1$*

$$a_{k+1} = a_k + \frac{a_k^2}{n}.$$

Prove that

$$1 - \frac{1}{n} < a_n < 1.$$

Solution. Note that

$$\frac{1}{a_{k+1}} = \frac{n}{na_k + a_k^2} = \frac{n}{a_k(n+a_k)} = \frac{1}{a_k} - \frac{1}{n+a_k}$$

thus

$$\frac{1}{a_k} - \frac{1}{a_{k+1}} = \frac{1}{n+a_k}.$$

We add up these relations for $k = 0, 1, \ldots, n-1$. We recognize a telescopic sum in the left-hand side, hence

$$2 - \frac{1}{a_n} = \sum_{k=0}^{n-1} \frac{1}{n+a_k}.$$

Each term $1/n + a_k$ appearing in the last sum is smaller than $1/n$, hence the whole sum is smaller than 1, which yields $2 - \dfrac{1}{a_n} < 1$ and then $a_n < 1$. Using again the recurrence relation, we obtain $a_{k+1} > a_k$, thus $a_k < 1$ for all $k \leq n$. But then

$$\sum_{k=0}^{n-1} \frac{1}{n+a_k} > \frac{n}{n+1}$$

and so

$$2 - \frac{1}{a_n} > \frac{n}{n+1}.$$

This simplifies to

$$a_n > \frac{n+1}{n+2} = 1 - \frac{1}{n+2} > 1 - \frac{1}{n},$$

which finishes the solution.

Example 1.26. (IMO Shortlist 2001) *Prove that for all real numbers $x_1, \ldots, x_n$ we have*

$$\frac{x_1}{1+x_1^2} + \frac{x_2}{1+x_1^2+x_2^2} + \cdots + \frac{x_n}{1+x_1^2+\cdots+x_n^2} < \sqrt{n}.$$

Solution. Using the Cauchy-Schwarz inequality we obtain

$$\left(\frac{x_1}{1+x_1^2}+\frac{x_2}{1+x_1^2+x_2^2}+\cdots+\frac{x_n}{1+x_1^2+\cdots+x_n^2}\right)^2$$

$$\leq n\cdot\sum_{k=1}^{n}\frac{x_k^2}{(1+x_1^2+\cdots+x_k^2)^2}.$$

Thus it suffices to prove that

$$\sum_{k=1}^{n}\frac{x_k^2}{(1+x_1^2+\cdots+x_k^2)^2}<1.$$

Letting $a_k=x_1^2+\cdots+x_k^2$ (with the convention that $a_0=0$), observe that

$$\begin{aligned}\frac{x_k^2}{(1+x_1^2+\cdots+x_k^2)^2}&=\frac{a_k-a_{k-1}}{(1+a_k)^2}\leq\frac{a_k-a_{k-1}}{(1+a_k)(1+a_{k-1})}\\&=\frac{1}{1+a_{k-1}}-\frac{1}{1+a_k}.\end{aligned}$$

We deduce that

$$\sum_{k=1}^{n}\frac{x_k^2}{(1+x_1^2+\cdots+x_k^2)^2}\leq\sum_{k=1}^{n}\left(\frac{1}{1+a_{k-1}}-\frac{1}{1+a_k}\right)=1-\frac{1}{1+a_n}<1,$$

as desired.

Example 1.27. *Prove that for all positive real numbers $x_1,x_2,\ldots,x_n$*

$$\frac{1}{1+x_1}+\frac{1}{1+x_1+x_2}+\cdots+\frac{1}{1+x_1+x_2+\cdots+x_n}<\sqrt{\frac{1}{x_1}+\frac{1}{x_2}+\cdots+\frac{1}{x_n}}.$$

Solution. This is very similar to the previous exercise. Using the Cauchy-Schwarz inequality yields

$$\left(\frac{1}{1+x_1}+\frac{1}{1+x_1+x_2}+\cdots+\frac{1}{1+x_1+x_2+\cdots+x_n}\right)^2$$

$$\leq \left(\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}\right)\left(\sum_{k=1}^{n} \frac{x_k}{(1+x_1+\cdots+x_k)^2}\right).$$

It suffices therefore to prove that

$$\sum_{k=1}^{n} \frac{x_k}{(1+x_1+\cdots+x_k)^2} < 1$$

for all positive real numbers $x_1, \ldots, x_n$. But this is precisely what has already been established during the solution of the previous exercise!

Example 1.28. *Prove that:*

$$2\left(\sqrt{n+1} - 1\right) < \sum_{k=1}^{n} \frac{1}{\sqrt{k}} < 2\sqrt{n}.$$

Solution. We have

$$\sqrt{k+1} - \sqrt{k} = \frac{1}{\sqrt{k+1} + \sqrt{k}} < \frac{1}{2\sqrt{k}}$$

because the last inequality is equivalent to $\sqrt{k+1} > \sqrt{k}$, and, similarly

$$\sqrt{k} - \sqrt{k-1} = \frac{1}{\sqrt{k} + \sqrt{k-1}} > \frac{1}{2\sqrt{k}}.$$

Thus,

$$2\left(\sqrt{k+1} - \sqrt{k}\right) < \frac{1}{\sqrt{k}} < 2\left(\sqrt{k} - \sqrt{k-1}\right)$$

for every positive integer k. We sum these inequalities for $k = 1, 2, \ldots, n$ and obtain

$$\sum_{k=1}^{n} 2\left(\sqrt{k+1} - \sqrt{k}\right) < \sum_{k=1}^{n} \frac{1}{\sqrt{k}} < \sum_{k=1}^{n} 2\left(\sqrt{k} - \sqrt{k-1}\right),$$

that is, the desired inequalities

$$2\left(\sqrt{n+1} - 1\right) < \sum_{k=1}^{n} \frac{1}{\sqrt{k}} < 2\sqrt{n},$$

because the first sum and the last sum telescope precisely to what we need.

Example 1.29. *Let $a_0 = 1$ and $a_{n+1} = a_0 \cdots a_n + 1$, $n \geq 0$. Prove that*

$$\frac{1}{a_1} + \cdots + \frac{1}{a_n} + \frac{1}{a_{n+1} - 1} = 1,$$

for all $n \geq 1$.

Solution. For $k \geq 1$ we have $a_0 \cdots a_{k-1} = a_k - 1$ and $a_0 \cdots a_{k-1} a_k = a_{k+1} - 1$. Then

$$a_{k+1} - 1 = (a_k - 1)a_k$$

and

$$\frac{1}{a_{k+1} - 1} = \frac{1}{(a_k - 1)a_k} = \frac{1}{a_k - 1} - \frac{1}{a_k},$$

meaning that

$$\frac{1}{a_k} = \frac{1}{a_k - 1} - \frac{1}{a_{k+1} - 1}.$$

Summing up from $k = 1$ to $k = n$ and noting that $a_1 = 2$ yields the desired result.

Example 1.30. (Romania TST 2003) *Let $(a_n)_{n \geq 1}$ be a sequence of real numbers given by $a_1 = 1/2$ and for each positive integer n*

$$a_{n+1} = \frac{a_n^2}{a_n^2 - a_n + 1}.$$

Prove that for every positive integer n we have $a_1 + a_2 + \cdots + a_n < 1$.

Solution. Setting $b_n = 1/a_n$, the recurrence relation becomes

$$b_{n+1} = b_n^2 - b_n + 1,$$

that is

$$\frac{b_{n+1} - 1}{b_n - 1} = b_n.$$

Multiplying these relations we get

$$b_{n+1} = b_n \cdots b_2 b_1 + 1 \Leftrightarrow \frac{1}{b_n} = \frac{1}{b_1 b_2 \cdots b_{n-1}} - \frac{1}{b_1 b_2 \cdots b_n}.$$

Telescoping again we get

$$a_1 + a_2 + \cdots + a_n = \frac{1}{b_1} + \frac{1}{b_2} + \cdots + \frac{1}{b_n} = 1 - \frac{1}{b_1 b_2 \cdots b_n} < 1$$

and we are done.
Can you see the similarities between this problem and the previous one?

Example 1.31. *Compute*

$$\frac{1}{1 \cdot 3} + \frac{2}{1 \cdot 3 \cdot 5} + \cdots + \frac{n}{1 \cdot 3 \cdot \ldots \cdot (2n+1)}.$$

Solution. Note that

$$\frac{k}{1 \cdot 3 \cdot \ldots \cdot (2k+1)} = \frac{1}{2} \cdot \frac{2k+1-1}{1 \cdot 3 \cdot \ldots \cdot (2k+1)}$$

$$= \frac{1}{2} \cdot \left(\frac{1}{1 \cdot 3 \cdot \ldots \cdot (2k-1)} - \frac{1}{1 \cdot 3 \cdot \ldots \cdot (2k+1)} \right).$$

We obtain therefore a telescopic sum, with value

$$\sum_{k=1}^{n} \frac{k}{1 \cdot 3 \cdot \ldots \cdot (2k+1)} = \frac{1}{2} \left(1 - \frac{1}{1 \cdot 3 \cdot \ldots \cdot (2n+1)} \right).$$

The "note that" part seems to come from nowhere, so let us explain a little bit more what is happening. Ideally we look for numbers a_k such that

$$\frac{k}{1 \cdot 3 \cdot \ldots \cdot (2k+1)} = a_k - a_{k+1}.$$

The form of the denominator strongly suggests taking

$$a_k = \frac{b_k}{1 \cdot 3 \cdot \ldots \cdot (2k-1)}$$

for some numbers b_k. The previous relation is then equivalent to

$$k = (2k+1)b_k - b_{k+1}.$$

Again, looking for a polynomial sequence $b_k = P(k)$ yields

$$X = (2X+1)P(X) - P(X+1)$$

and degree considerations show that P is constant, necessarily equal to $1/2$.
As shown in the introduction the idea in telescoping products is very similar to that related to sums (only cancellations are between multiplicative factors rather than additive terms). We repeatedly use the following result:

$$\prod_{k=1}^{n} \frac{a_{k+1}}{a_k} = \frac{a_{n+1}}{a_1},$$

where $a_k \neq 0$ for all k.
We have already seen some simple examples in the introductory part. Let us see some more.

Example 1.32. *Evaluate the product*

$$\prod_{k=1}^{n} \left(1 + \frac{2^k}{1+2^k}\right).$$

Solution. Things are fairly simple here, since

$$1 + \frac{2^k}{1+2^k} = \frac{1+2^k+2^k}{1+2^k} = \frac{1+2^{k+1}}{1+2^k},$$

the product is telescopic and

$$\prod_{k=1}^{n} \left(1 + \frac{2^k}{1+2^k}\right) = \prod_{k=1}^{n} \left(\frac{1+2^{k+1}}{1+2^k}\right) = \frac{1+2^{n+1}}{3}.$$

Example 1.33. *Evaluate*

$$\frac{\left(1^4 + \frac{1}{4}\right)\left(3^4 + \frac{1}{4}\right)\cdots\left((2n-1)^4 + \frac{1}{4}\right)}{\left(2^4 + \frac{1}{4}\right)\left(4^4 + \frac{1}{4}\right)\cdots\left((2n)^4 + \frac{1}{4}\right)}.$$

Solution. By the formula

$$a^4+\frac{1}{4}=\left(a^2-a+\frac{1}{2}\right)\left(a^2+a+\frac{1}{2}\right)$$
$$=\left(a^2-a+\frac{1}{2}\right)\left((a+1)^2-(a+1)+\frac{1}{2}\right),$$

we have, after all simplifications, that our product is equal to

$$\frac{\prod_{j=1}^{n}\left((2j-1)^4+\frac{1}{4}\right)}{\prod_{j=1}^{n}\left((2j)^4+\frac{1}{4}\right)}=\frac{\prod_{j=1}^{n}\left((2j-1)^2-(2j-1)+\frac{1}{2}\right)\left((2j)^2-(2j)+\frac{1}{2}\right)}{\prod_{j=1}^{n}\left((2j)^2-(2j)+\frac{1}{2}\right)\left((2j+1)^2-(2j+1)+\frac{1}{2}\right)}$$
$$=\frac{\frac{1}{2}}{(2n+1)^2-(2n+1)+\frac{1}{2}}=\frac{1}{8n^2+4n+1}.$$

Example 1.34. *Evaluate*

$$\prod_{k=2}^{n}\left(\frac{k^3-1}{k^3+1}\right).$$

Solution. The numbers k^3-1 and k^3+1 strongly suggest the use of the classical formulae

$$a^3-b^3=(a-b)(a^2+ab+b^2),\quad a^3+b^3=(a+b)(a^2-ab+b^2).$$

We deduce that

$$\frac{k^3-1}{k^3+1}=\frac{(k-1)(k^2+k+1)}{(k+1)(k^2-k+1)}=\frac{k-1}{k+1}\cdot\frac{k^2+k+1}{k^2-k+1}.$$

Note that $\prod_{k=2}^{n}\frac{k-1}{k+1}$ is a telescopic product:

$$\prod_{k=2}^{n}\frac{k-1}{k+1}=\prod_{k=2}^{n}\frac{k(k-1)}{k(k+1)}=\frac{2\cdot 1}{n(n+1)}.$$

The key observation is that

$$k^2 + k + 1 = (k+1)^2 - (k+1) + 1,$$

thus $\displaystyle\prod_{k=2}^{n} \frac{k^2+k+1}{k^2-k+1}$ is also a telescopic product, more precisely

$$\prod_{k=2}^{n} \frac{k^2+k+1}{k^2-k+1} = \prod_{k=2}^{n} \frac{(k+1)^2-(k+1)+1}{k^2-k+1} = \frac{(n+1)^2-(n+1)+1}{3}.$$

Putting everything together we obtain

$$\prod_{k=2}^{n} \left(\frac{k^3-1}{k^3+1}\right) = \frac{2}{n(n+1)} \cdot \frac{(n+1)^2-(n+1)+1}{3} = \frac{2(n^2+n+1)}{3n(n+1)}.$$

Again, this allows evaluating the corresponding infinite product:

$$\prod_{k=2}^{\infty} \left(\frac{k^3-1}{k^3+1}\right) = \lim_{n\to\infty} \prod_{k=2}^{n} \left(\frac{k^3-1}{k^3+1}\right) = \lim_{n\to\infty} \frac{2(n^2+n+1)}{3n(n+1)} = \frac{2}{3}.$$

Example 1.35. *Prove that for all* $n \geq 1$

$$\prod_{k=1}^{n} \left(1+\frac{1}{k^3}\right) < 3.$$

Solution. We have for $n \geq 2$

$$\prod_{k=1}^{n} \left(1+\frac{1}{k^3}\right) = 2\prod_{k=2}^{n} \frac{k^3+1}{k^3} < 2\prod_{k=2}^{n} \frac{k^3+1}{k^3-1}.$$

In the previous problem we have already computed the last product (actually its inverse), which equals $\dfrac{3(n^2+n)}{2(n^2+n+1)}$, thus

$$\prod_{k=1}^{n} \left(1+\frac{1}{k^3}\right) < 2 \cdot \frac{3(n^2+n)}{2(n^2+n+1)} = 3 \cdot \frac{n^2+n}{n^2+n+1} < 3.$$

We can also use induction to show the stronger inequality

$$\prod_{k=1}^{n}\left(1+\frac{1}{k^3}\right) \le 3-\frac{1}{n}$$

(sometimes a stronger inequality can be proven by the inductive method, while the weaker one cannot be thus proven). Actually the equal sign holds only for $n=1$ (which is immediately seen) for $n \ge 2$ the inequality is strict. We still need to prove that if

$$\prod_{k=1}^{n}\left(1+\frac{1}{k^3}\right) \le 3-\frac{1}{n},$$

then

$$\prod_{k=1}^{n+1}\left(1+\frac{1}{k^3}\right) \le 3-\frac{1}{n+1}.$$

And, indeed, by using the induction hypothesis, we have

$$\begin{aligned}\prod_{k=1}^{n+1}\left(1+\frac{1}{k^3}\right) &= \left(\prod_{k=1}^{n}\left(1+\frac{1}{k^3}\right)\right)\left(1+\frac{1}{(n+1)^3}\right)\\ &\le \left(3-\frac{1}{n}\right)\left(1+\frac{1}{(n+1)^3}\right) < 3-\frac{1}{n+1},\end{aligned}$$

the last inequality being equivalent to

$$\left(3-\frac{1}{n}\right)\frac{1}{(n+1)^3} < \frac{1}{n}-\frac{1}{n+1} \Leftrightarrow \frac{3n-1}{n(n+1)^3} < \frac{1}{n(n+1)}$$

$$\Leftrightarrow 3n-1 < (n+1)^2 \Leftrightarrow 0 < n^2-n+2.$$

Example 1.36. *Prove that for any positive integer $n > 1$ we have*

$$\frac{1}{2\sqrt{n}} < \frac{1}{2}\frac{3}{4}\cdots\frac{2n-1}{2n} < \frac{1}{\sqrt{2n+1}}.$$

Solution. We have

$$\left(\prod_{k=1}^{n} \frac{2k-1}{2k}\right)^2 = \frac{1}{2}\left(\prod_{k=2}^{n} \frac{(2k-1)^2}{(2k-2)(2k)}\right)\frac{1}{2n} > \frac{1}{4n},$$

whence the first inequality follows. Also,

$$\left(\prod_{k=1}^{n} \frac{2k-1}{2k}\right)^2 = \left(\prod_{k=1}^{n} \frac{(2k-1)(2k+1)}{(2k)^2}\right)\frac{1}{2n+1} < \frac{1}{2n+1},$$

which gives the second inequality. We used

$$\frac{(2k-1)^2}{(2k-2)(2k)} = \frac{4k^2-4k+1}{4k^2-4k} > 1, \ k \geq 2,$$

and

$$\frac{(2k-1)(2k+1)}{(2k)^2} = \frac{4k^2-1}{4k^2} < 1, \ k \geq 1.$$

Example 1.37. *Let $a_n = 1+2+3+\cdots+n$. Compute*

$$\frac{a_2}{a_2-1} \cdot \frac{a_3}{a_3-1} \cdots \frac{a_n}{a_n-1}.$$

Solution. We have

$$a_n = \frac{n(n+1)}{2},$$

hence

$$\frac{a_k}{a_k-1} = \frac{k(k+1)}{k(k+1)-2}.$$

The key remark is that the denominator factors nicely, since

$$k(k+1)-2 = k^2+k-2 = (k-1)(k+2).$$

Thus

$$\frac{a_k}{a_k-1} = \frac{k(k+1)}{(k-1)(k+2)} = \frac{k}{k-1} \cdot \frac{k+1}{k+2}.$$

We recognize two telescopic products and deduce that

$$\prod_{k=2}^{n} \frac{a_k}{a_k-1} = \prod_{k=2}^{n} \frac{k}{k-1} \cdot \prod_{k=2}^{n} \frac{k+1}{k+2} = \frac{n}{1} \cdot \frac{3}{n+2} = \frac{3n}{n+2}.$$

Chapter 2

Telescoping Sums and Products in Trigonometry

To solve this type of problems you need to have a good grasp of trigonometric identities. These identities are studied in school, but we recall some of them here.

Half-angle formulas

$$\sin a = \frac{2\tan\dfrac{a}{2}}{1+\tan^2\dfrac{a}{2}},$$

$$\cos a = \frac{1-\tan^2\dfrac{a}{2}}{1+\tan^2\dfrac{a}{2}},$$

$$\tan a = \frac{2\tan\dfrac{a}{2}}{1-\tan^2\dfrac{a}{2}}.$$

Triple-angle formulas

$$\sin 3a = 3\sin a - 4\sin^3 a,$$
$$\cos 3a = 4\cos^3 a - 3\cos a,$$
$$\tan 3a = \frac{3\tan a - \tan^3 a}{1 - 3\tan^2 a}.$$

Sum-to-product formulas

$$\sin a + \sin b = 2\sin\frac{a+b}{2}\cos\frac{a-b}{2},$$
$$\cos a + \cos b = 2\cos\frac{a+b}{2}\cos\frac{a-b}{2},$$
$$\tan a + \tan b = \frac{\sin(a+b)}{\cos a\cos b},$$
$$\sin a - \sin b = 2\sin\frac{a-b}{2}\cos\frac{a+b}{2},$$
$$\cos a - \cos b = -2\sin\frac{a-b}{2}\sin\frac{a+b}{2},$$
$$\tan a - \tan b = \frac{\sin(a-b)}{\cos a\cos b}.$$

Addition-subtraction formulas for tangent

$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta},$$
$$\tan(\alpha-\beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}.$$

Example 2.1. *Prove that for all real numbers x, y, z we have*

(a) $\sin x + \sin y + \sin z - \sin(x+y+z) = 4\sin\frac{x+y}{2}\sin\frac{x+z}{2}\sin\frac{y+z}{2}$;

(b) $\cos x + \cos y + \cos z + \cos(x+y+z) = 4\cos\frac{x+y}{2}\cos\frac{x+z}{2}\cos\frac{y+z}{2}$.

Solution. (a) Applying the formula

$$\sin a - \sin b = 2\sin\frac{a-b}{2}\cos\frac{a+b}{2}$$

we obtain

$$\sin x - \sin(x+y+z) = -2\sin\frac{y+z}{2}\cos\left(x+\frac{y+z}{2}\right).$$

On the other hand,

$$\sin y + \sin z = 2\sin\frac{y+z}{2}\cos\frac{y-z}{2}.$$

Thus

$$\sin x+\sin y+\sin z-\sin(x+y+z) = 2\sin\frac{y+z}{2}\left(\cos\frac{y-z}{2} - \cos\left(x+\frac{y+z}{2}\right)\right).$$

It suffices therefore to prove that

$$\cos\frac{y-z}{2} - \cos\left(x+\frac{y+z}{2}\right) = 2\sin\frac{x+y}{2}\sin\frac{x+z}{2},$$

which is another consequence of the formula

$$\cos a - \cos b = 2\sin\frac{b-a}{2}\sin\frac{a+b}{2}.$$

For part (b) we can proceed similarly, or we can transform products into sums by using the formula

$$2\cos a\cos b = \cos(a-b) + \cos(a+b);$$

so, we have

$$\begin{aligned}
4\cos\frac{x+y}{2}\cos\frac{x+z}{2}\cos\frac{y+z}{2} &= 2\left(\cos\frac{2x+y+z}{2} + \cos\frac{y-z}{2}\right)\cos\frac{y+z}{2}\\
&= 2\cos\frac{2x+y+z}{2}\cos\frac{y+z}{2} + 2\cos\frac{y-z}{2}\cos\frac{y+z}{2}\\
&= \cos(x+y+z) + \cos x + \cos y + \cos z.
\end{aligned}$$

The attentive reader has probably already noted that any of these identities transforms into the other one under the substitutions $x \mapsto \frac{\pi}{2} - x$, $y \mapsto \frac{\pi}{2} - y$, $z \mapsto \frac{\pi}{2} - z$. (So, basically, we have a single identity with two forms.)

Example 2.2. *Prove that in any triangle ABC we have*

$$\cos\frac{A}{2} + \cos\frac{B}{2} + \cos\frac{C}{2} = 4\sin\frac{\pi + A}{4}\sin\frac{\pi + B}{4}\sin\frac{\pi + C}{4}.$$

Solution. Note that since $A + B + C = 180^\circ$, we have

$$\cos\frac{A}{2} = \cos\left(90^\circ - \frac{B+C}{2}\right) = \sin\frac{B+C}{2}$$

and similarly for $\cos\frac{B}{2}$ and $\cos\frac{C}{2}$. Using Example 2.1 we obtain

$$\cos\frac{A}{2} + \cos\frac{B}{2} + \cos\frac{C}{2} - \sin(A+B+C)$$

$$= 4\sin\frac{(A+B)+(B+C)}{4}\sin\frac{(B+C)+(C+A)}{4}\sin\frac{(C+A)+(A+B)}{4}.$$

On the other hand, $\sin(A+B+C) = 0$ and

$$(A+B)+(B+C) = A+B+C+B = \pi + B,$$

yielding the desired result.

Example 2.3. *Consider two triangles ABC and $A'B'C'$. Prove that*

$$\sin(A+A') + \sin(B+B') + \sin(C+C') = 4\sin\frac{A+A'}{2}\sin\frac{B+B'}{2}\sin\frac{C+C'}{2}.$$

Solution. Note that

$$(A+A') + (B+B') + (C+C') = 360^\circ,$$

thus using Example 2.1 we can write

$$\sin(A+A') + \sin(B+B') + \sin(C+C')$$

$$= 4\sin\frac{A+A'+B+B'}{2}\sin\frac{B+B'+C+C'}{2}\sin\frac{C+C'+A+A'}{2}.$$

On the other hand

$$\frac{A+A'+B+B'}{2}=\frac{360^\circ-(C+C')}{2}=180^\circ-\frac{C+C'}{2},$$

hence

$$\sin\frac{A+A'+B+B'}{2}=\sin\frac{C+C'}{2}.$$

Similarly we obtain

$$\sin\frac{B+B'+C+C'}{2}=\sin\frac{A+A'}{2},\quad \sin\frac{C+C'+A+A'}{2}=\sin\frac{B+B'}{2}$$

and the result follows.

Example 2.4. *Prove that*

$$(4\cos^2 9^\circ-1)(4\cos^2 27^\circ-1)(4\cos^2 81^\circ-1)(4\cos^2 243^\circ-1)$$

is an integer.

Solution. The formula

$$\sin 3x = 4\cos^2 x\sin x-\sin x$$

can be easily obtained by the usual transformations:

$$\begin{aligned}\sin 3x&=\sin(2x+x)=\sin 2x\cos x+\sin x\cos 2x\\&=2\sin x\cos^2 x+\sin x(2\cos^2 x-1)\\&=4\cos^2 x\sin x-\sin x\end{aligned}$$

(it is one of the forms in which $\sin 3x$ can appear). Thus we have

$$4\cos^2 x-1=\frac{\sin 3x}{\sin x}$$

and our product becomes

$$\begin{aligned}&(4\cos^2 9^\circ-1)(4\cos^2 27^\circ-1)(4\cos^2 81^\circ-1)(4\cos^2 243^\circ-1)\\&=\frac{\sin 27^\circ}{\sin 9^\circ}\frac{\sin 81^\circ}{\sin 27^\circ}\frac{\sin 243^\circ}{\sin 81^\circ}\frac{\sin 729^\circ}{\sin 243^\circ}=\frac{\sin 729^\circ}{\sin 9^\circ}=1,\end{aligned}$$

because $\sin 729^\circ=\sin(9^\circ+2\cdot 360^\circ)=\sin 9^\circ$.

Example 2.5. *Prove that*

$$(4\cos^2 9^\circ - 3)(4\cos^2 27^\circ - 3) = \tan 9^\circ.$$

Solution. We use the formula (prove it!)

$$4\cos^2 x - 3 = \frac{\cos 3x}{\cos x},$$

which yields

$$(4\cos^2 9^\circ - 3)(4\cos^2 27^\circ - 3) = \frac{\cos 27^\circ}{\cos 9^\circ} \cdot \frac{\cos 81^\circ}{\cos 27^\circ} = \frac{\cos 81^\circ}{\cos 9^\circ}.$$

On the other hand, we have

$$\cos 81^\circ = \sin(90 - 81)^\circ = \sin 9^\circ,$$

which finishes the proof.

Example 2.6. *Prove that*

$$\sin^2 18^\circ + \sin^2 30^\circ = \sin^2 36^\circ.$$

Solution. Using the formulae

$$\sin^2(x) = \frac{1-\cos(2x)}{2}, \quad \cos a - \cos b = 2\sin\frac{b-a}{2}\sin\frac{a+b}{2},$$

we obtain

$$\sin^2 36^\circ - \sin^2 18^\circ = \frac{\cos 36^\circ - \cos 72^\circ}{2} = \sin 18^\circ \sin 54^\circ.$$

Since $\sin^2 30^\circ = \frac{1}{4}$, it suffices to prove the equality

$$\sin 18^\circ \sin 54^\circ = \frac{1}{4}.$$

On the other hand

$$\begin{aligned} 4\sin 18^\circ \sin 54^\circ \cos 18^\circ &= 2\sin 36^\circ \sin 54^\circ \\ &= 2\sin 36^\circ \cos 36^\circ \\ &= \sin 72^\circ = \cos 18^\circ. \end{aligned}$$

Dividing by the nonzero number $\cos 18^\circ$ yields the desired equality.

Example 2.7. *Prove that*

$$\tan^2 36^\circ \tan^2 72^\circ = 5.$$

Solution. Note that, basically, all values of the trigonometric functions for arguments 18°, 36°, 54°, or 72° can be written as simple algebraic expressions. Specifically, we have

$$\cos 36^\circ = \sin 54^\circ = \frac{\sqrt{5}+1}{4}, \quad \cos 54^\circ = \sin 36^\circ = \frac{\sqrt{10-2\sqrt{5}}}{4},$$

and

$$\cos 72^\circ = \sin 18^\circ = \frac{\sqrt{5}-1}{4}, \quad \cos 18^\circ = \sin 72^\circ = \frac{\sqrt{10+2\sqrt{5}}}{4}.$$

For instance, if we denote by $t = \cos 36^\circ$, from the equality

$$\sin(2 \cdot 36^\circ) = \sin(3 \cdot 36^\circ)$$

we infer

$$2t \sin 36^\circ = (4t^2 - 1) \sin 36^\circ$$

hence

$$4t^2 - 2t - 1 = 0.$$

As $t > 0$ we get $t = (\sqrt{5}+1)/4$, and all the other values follow from this by basic formulas. Of course, this observation provides another solution for the previous exercise. Now, for the present one, we have

$$\begin{aligned}\tan 36^\circ \tan 72^\circ &= \frac{\sin 36^\circ}{\cos 36^\circ} \cdot \frac{\sin 72^\circ}{\cos 72^\circ} = \frac{2\sin^2 36^\circ}{\cos 72^\circ} \\ &= \frac{2 - 2\cos^2 36^\circ}{2\cos^2 36^\circ - 1} = \sqrt{5},\end{aligned}$$

because, as we have seen, $\cos 36^\circ = \dfrac{\sqrt{5}+1}{4}$.

Example 2.8. *Compute*

$$\frac{1}{1+\cot 1^\circ}+\frac{1}{1+\cot 5^\circ}+\frac{1}{1+\cot 9^\circ}+\cdots+\frac{1}{1+\cot 89^\circ}.$$

Solution. The key observation is that if $x+y=90^\circ$, then

$$\frac{1}{1+\cot x}+\frac{1}{1+\cot y}=1.$$

Indeed, we have

$$\cot(x)=\cot(90^\circ-y)=\tan y,$$

hence

$$\begin{aligned}\frac{1}{1+\cot x}+\frac{1}{1+\cot y}&=\frac{1}{1+\tan y}+\frac{1}{1+\dfrac{1}{\tan y}}\\&=\frac{1}{1+\tan y}+\frac{\tan y}{1+\tan y}=1.\end{aligned}$$

It follows that

$$\frac{1}{1+\cot 1^\circ}+\frac{1}{1+\cot 89^\circ}=1,\quad \frac{1}{1+\cot 5^\circ}+\frac{1}{1+\cot 84^\circ}=1,\ldots$$

Adding these relations and taking into account the term $\dfrac{1}{1+\cot 45^\circ}$ (which cannot be paired with another term), we obtain

$$\begin{aligned}\frac{1}{1+\cot 1^\circ}+\frac{1}{1+\cot 5^\circ}+\frac{1}{1+\cot 9^\circ}+\cdots+\frac{1}{1+\cot 89^\circ}&=11+\frac{1}{1+\cot 45^\circ}\\&=11+\frac{1}{2}=11.5.\end{aligned}$$

Example 2.9. *Prove that*

$$\tan 10^\circ=\tan 20^\circ\cdot\tan 30^\circ\cdot\tan 40^\circ.$$

Solution. Using the formula

$$\tan x = \frac{\sin x}{\cos x},$$

the equality is equivalent to

$$\sin 10^\circ \cos 20^\circ \cos 30^\circ \cos 40^\circ = \cos 10^\circ \sin 20^\circ \sin 30^\circ \sin 40^\circ.$$

We now use the identities

$$\cos 20^\circ \cos 40^\circ = \frac{\cos 60^\circ + \cos 20^\circ}{2}$$

and

$$\sin 20^\circ \sin 40^\circ = \frac{\cos 20^\circ - \cos 60^\circ}{2}$$

to reduce the previous equality to

$$\sin 10^\circ \cos 30^\circ (\cos 60^\circ + \cos 20^\circ) = \sin 30^\circ \cos 10^\circ (\cos 20^\circ - \cos 60^\circ),$$

which is equivalent to

$$\cos 60^\circ (\sin 10^\circ \cos 30^\circ + \sin 30^\circ \cos 10^\circ) = \cos 20^\circ (\sin 30^\circ \cos 10^\circ - \sin 10^\circ \cos 30^\circ).$$

Using again the standard formulae, this reduces to

$$\cos 60^\circ \sin 40^\circ = \cos 20^\circ \sin 20^\circ.$$

Since $\cos 60^\circ = 1/2$, the previous equality reduces to the standard formula

$$2 \sin x \cos x = \sin(2x) \text{ for } x = 20^\circ.$$

Example 2.10. *Prove that*

$$\frac{1}{2} + \cos\frac{2\pi}{11} + \cos\frac{4\pi}{11} = \frac{1}{4\sin\frac{\pi}{22}}.$$

Solution. Using the formula

$$2\sin x\cos x = \sin(2x),$$

we can write the right-hand side

$$\frac{1}{4\sin\dfrac{\pi}{22}} = \frac{\cos\dfrac{\pi}{22}}{2\sin\dfrac{\pi}{11}},$$

hence multiplying everything by $2\sin\dfrac{\pi}{11}$ we are reduced to proving the equivalent equality

$$\sin\frac{\pi}{11} + 2\sin\frac{\pi}{11}\cos\frac{2\pi}{11} + 2\sin\frac{\pi}{11}\cos\frac{4\pi}{11} = \cos\frac{\pi}{22}.$$

On the other hand, we have

$$2\sin\frac{\pi}{11}\cos\frac{2\pi}{11} = \sin\frac{3\pi}{11} - \sin\frac{\pi}{11}$$

and

$$2\sin\frac{\pi}{11}\cos\frac{4\pi}{11} = \sin\frac{5\pi}{11} - \sin\frac{3\pi}{11},$$

thus

$$\begin{aligned}&\sin\frac{\pi}{11} + 2\sin\frac{\pi}{11}\cos\frac{2\pi}{11} + 2\sin\frac{\pi}{11}\cos\frac{4\pi}{11}\\ &= \sin\frac{\pi}{11} + \sin\frac{3\pi}{11} - \sin\frac{\pi}{11} + \sin\frac{5\pi}{11} - \sin\frac{3\pi}{11} = \sin\frac{5\pi}{11}.\end{aligned}$$

All in all, it suffices to prove the equality

$$\sin\frac{5\pi}{11} = \cos\frac{\pi}{22},$$

which follows from

$$\sin\frac{5\pi}{11} = \cos\left(\frac{\pi}{2} - \frac{5\pi}{11}\right) = \cos\frac{\pi}{22}.$$

Example 2.11. *Prove that for all $x \in \mathbb{R}$*

$$\sin x \sin(60^\circ - x) \sin(60^\circ + x) = \frac{1}{4} \sin 3x$$

and

$$\cos x \cos(60^\circ - x) \cos(60^\circ + x) = \frac{1}{4} \cos 3x.$$

Solution. Since

$$2 \sin(60^\circ - x) \sin(60^\circ + x) = \cos 2x - \cos(120^\circ) = \cos 2x + \frac{1}{2},$$

it suffices to prove that

$$2 \sin x \left(\cos 2x + \frac{1}{2} \right) = \sin 3x,$$

which is equivalent to

$$\sin 3x - \sin x = 2 \sin x \cos 2x,$$

another consequence of the standard identity

$$\sin a - \sin b = 2 \sin \frac{a-b}{2} \cos \frac{a+b}{2}.$$

For the second identity we present a slightly different approach. By the usual formulae for the cosine of the sum and the difference we have

$$\begin{aligned} \cos(60^\circ - x) \cos(60^\circ + x) &= \cos^2 60^\circ \cos^2 x - \sin^2 60^\circ \sin^2 x \\ &= \frac{1}{4}(\cos^2 x - 3 \sin^2 x). \end{aligned}$$

On the other hand,

$$\begin{aligned} \cos 3x = \cos(2x + x) &= \cos 2x \cos x - \sin 2x \sin x \\ &= (\cos^2 x - \sin^2 x) - 2 \sin^2 x \cos x \\ &= \cos x(\cos^2 x - 3 \sin^2 x), \end{aligned}$$

therefore,

$$\cos x \cos(60^\circ - x) \cos(60^\circ + x) = \frac{1}{4} \cos x(\cos^2 x - 3 \sin^2 x) = \frac{1}{4} \cos 3x.$$

Example 2.12. *Compute*

$$\sin 10^\circ \sin 50^\circ \sin 70^\circ.$$

Solution. We have

$$\begin{aligned}\sin 10^\circ \sin 50^\circ \sin 70^\circ &= \cos 80^\circ \cos 40^\circ \cos 20^\circ \\ &= \frac{1}{\sin 20^\circ} \sin 20^\circ \cos 20^\circ \cos 40^\circ \cos 80^\circ \\ &= \frac{1}{\sin 20^\circ} \frac{1}{2} \sin 40^\circ \cos 40^\circ \cos 80^\circ \\ &= \frac{1}{4 \sin 20^\circ} \sin 80^\circ \cos 80^\circ \\ &= \frac{1}{8 \sin 20^\circ} \sin 160^\circ = \frac{1}{8}.\end{aligned}$$

Of course, this is just an application of the first part of the previous example, for $x = 10^\circ$. Apply the second part to the same $x = 10^\circ$ and try to find a different solution.

Example 2.13. *Prove that*

$$\cos 6^\circ \cos 42^\circ \cos 66^\circ \cos 78^\circ = \frac{1}{16}.$$

Solution. By the second identity from the previous exercise (applied for $x = 6^\circ$ and $x = 18^\circ$ respectively) we have

$$\cos 6^\circ \cos 54^\circ \cos 66^\circ = \frac{1}{4} \cos 18^\circ$$

and

$$\cos 18^\circ \cos 42^\circ \cos 78^\circ = \frac{1}{4} \cos 54^\circ.$$

Clearly it suffices now to multiply side by side the two above equalities (and then simplify by cancelling $\cos 18^\circ \cos 54^\circ \neq 0$) in order to get the desired result.

Example 2.14. *Prove that*

$$\sin 5^\circ \sin 15^\circ \sin 25^\circ \cdots \sin 85^\circ = \frac{1}{256\sqrt{2}}.$$

Solution. We have

$$\sin 5^\circ \sin 15^\circ \sin 25^\circ \cdots \sin 85^\circ = (\sin 5^\circ \sin 85^\circ)(\sin 15^\circ \sin 75^\circ) \cdots \sin 45^\circ$$

$$= (\sin 5^\circ \cos 5^\circ)(\sin 15^\circ \cos 15^\circ)(\sin 25^\circ \cos 25^\circ)(\sin 35^\circ \cos 35^\circ) \sin 45^\circ$$

$$= \frac{1}{16\sqrt{2}} \sin 10^\circ \sin 30^\circ \sin 50^\circ \sin 70^\circ$$

$$= \frac{1}{32\sqrt{2}} \sin 10^\circ \sin 50^\circ \sin 70^\circ.$$

Applying Example 2.14 yields the desired result.

Example 2.15. *Prove that*

$$S = \tan^2 10^\circ + \tan^2 50^\circ + \tan^2 70^\circ = 9.$$

Solution. More generally, we will show that

$$S = \tan^2 x + \tan^2(60^\circ - x) + \tan^2(60^\circ + x) = 6 + 9\tan^2 3x.$$

Let $\tan x = t$, then

$$\tan(60^\circ - x) = \frac{\sqrt{3} - t}{1 + \sqrt{3}t}$$

and

$$\tan(60^\circ + x) = \frac{\sqrt{3} + t}{1 - \sqrt{3}t}.$$

Consequently

$$\begin{aligned}
S &= t^2 + \left(\frac{\sqrt{3}-t}{1+\sqrt{3}t}\right)^2 + \left(\frac{\sqrt{3}+t}{1-\sqrt{3}t}\right)^2 = \frac{6+45t^2+9t^6}{(1-3t^2)^2} \\
&= \frac{6(1-6t^2+9t^4)+81t^2-54t^4+9t^6}{(1-3t^2)^2} \\
&= 6 + \frac{81t^2-54t^4+9t^6}{(1-3t^2)^2} = 6+9\cdot\frac{9t^2-6t^4+t^6}{(1-3t^2)^2} \\
&= 6+9\cdot\frac{(3t-t^3)^2}{(1-3t^2)^2} = 6+9\tan^2 3x.
\end{aligned}$$

For $x = 10^\circ$ we get

$$S = 6 + 9\tan^2 30^\circ = 6 + 9\cdot\frac{1}{3} = 9,$$

as desired.

Example 2.16. *Prove that*

$$\tan 50^\circ + \tan 60^\circ + \tan 70^\circ = \tan 80^\circ.$$

Solution. First verify that, for $a+b+c = 180^\circ$ (that is, for example, for the measures of the angles of a triangle), we have

$$\tan a + \tan b + \tan c = \tan a \tan b \tan c$$

(start with $\tan c = -\tan(a+b)$). In our case,

$$\begin{aligned}
\tan 50^\circ + \tan 60^\circ + \tan 70^\circ &= \tan 50^\circ \tan 60^\circ \tan 70^\circ \\
&= \tan 60^\circ \tan(60^\circ - 10^\circ)\tan(60^\circ + 10^\circ) \\
&= \tan 60^\circ \cdot \frac{\sqrt{3}-\tan 10^\circ}{1+\sqrt{3}\tan 10^\circ}\cdot\frac{\sqrt{3}+\tan 10^\circ}{1-\sqrt{3}\tan 10^\circ} \\
&= \frac{1}{\tan 30^\circ}\cdot\frac{1}{\tan 10^\circ}\cdot\frac{3\tan 10^\circ - \tan^3 10^\circ}{1-3\tan^2 10^\circ} \\
&= \frac{1}{\tan 10^\circ} = \cot 10^\circ = \tan 80^\circ.
\end{aligned}$$

We used again the formula for the tangent of the triple of an angle

$$\tan 3x = \frac{3\tan x - \tan^3 x}{1 - 3\tan^2 x},$$

for $x = 10°$.

Example 2.17. *Prove that*

$$\sin 25° \sin 35° \sin 60° \sin 85° = \sin 20° \sin 40° \sin 75° \sin 80°.$$

Solution. Using Example 2.12, we obtain

$$\sin 25° \sin 35° \sin 85° = \sin 25° \sin(60° - 25°)\sin(60° + 25°) = \frac{1}{4}\sin 75°.$$

Similarly, we have

$$\sin 20° \sin 40° \sin 80° = \frac{1}{4}\sin 60°.$$

It follows that both products are equal to $\frac{1}{4}\sin 60° \sin 75°$, in particular they are equal.

Example 2.18. *Show that*

$$S = \cos\frac{\pi}{7} - \cos\frac{2\pi}{7} + \cos\frac{3\pi}{7} = \frac{1}{2}.$$

Solution. We have (by transforming the products into sums)

$$2S\sin\frac{\pi}{7} = \sin\frac{2\pi}{7} - \sin\frac{3\pi}{7} + \sin\frac{\pi}{7} + \sin\frac{4\pi}{7} - \sin\frac{2\pi}{7} = \sin\frac{\pi}{7},$$

because

$$\sin\frac{3\pi}{7} = \sin\left(\pi - \frac{3\pi}{7}\right) = \sin\frac{4\pi}{7}.$$

The result follows after dividing by $\sin\frac{\pi}{7}$, which is nonzero.

Example 2.19. *Prove that*

$$\cot^2\frac{\pi}{7}+\cot^2\frac{2\pi}{7}+\cot^2\frac{3\pi}{7}=5.$$

Solution. Using

$$\cot 2\alpha=\frac{\cot^2\alpha-1}{2\cot\alpha}\Rightarrow\cot^2\alpha=2\cot\alpha\cot 2\alpha+1$$

and

$$\cot(\pi-\alpha)=-\cot\alpha,$$

we get

$$\begin{aligned}\cot^2\frac{\pi}{7}&=2\cot\frac{\pi}{7}\cot\frac{2\pi}{7}+1\\ \cot^2\frac{2\pi}{7}&=-2\cot\frac{2\pi}{7}\cot\frac{3\pi}{7}+1\\ \cot^2\frac{3\pi}{7}&=-2\cot\frac{3\pi}{7}\cot\frac{\pi}{7}+1.\end{aligned}$$

Summing those up, we get

$$\cot^2\frac{\pi}{7}+\cot^2\frac{2\pi}{7}+\cot^2\frac{3\pi}{7}=3+2\left(\cot\frac{\pi}{7}\cot\frac{2\pi}{7}-\cot\frac{3\pi}{7}\left(\cot\frac{\pi}{7}+\cot\frac{2\pi}{7}\right)\right).$$

Then, using

$$\cot\alpha+\cot\beta=\frac{\sin(\alpha+\beta)}{\sin\alpha\sin\beta},$$

the sum becomes

$$\begin{aligned}\cot^2\frac{\pi}{7}+\cot^2\frac{2\pi}{7}+\cot^2\frac{3\pi}{7}&=3+2\left(\cot\frac{\pi}{7}\cot\frac{2\pi}{7}-\cot\frac{3\pi}{7}\cdot\frac{\sin\frac{3\pi}{7}}{\sin\frac{\pi}{7}\sin\frac{2\pi}{7}}\right)\\ &=3+2\frac{\cos\frac{\pi}{7}\cos\frac{2\pi}{7}-\cos\frac{3\pi}{7}}{\sin\frac{\pi}{7}\sin\frac{2\pi}{7}}\end{aligned}$$

$$= 3 + 2\frac{\cos\frac{\pi}{7}\cos\frac{2\pi}{7} - \cos\left(\frac{\pi}{7}+\frac{2\pi}{7}\right)}{\sin\frac{\pi}{7}\sin\frac{2\pi}{7}} = 3 + 2\frac{\sin\frac{\pi}{7}\sin\frac{2\pi}{7}}{\sin\frac{\pi}{7}\sin\frac{2\pi}{7}} = 5.$$

Note that this equality can be seen as an instance of the general formula

$$\sum_{k=1}^{n}\cot^2\frac{k\pi}{2n+1} = \frac{\binom{2n+1}{3}}{\binom{2n+1}{1}} = \frac{n(2n-1)}{3}$$

(for $n = 3$ we get our result). This general formula holds because as we will see in Example 3.5 in the next chapter

$$\cot^2\frac{\pi}{2n+1},\ \cot^2\frac{2\pi}{2n+1},\ldots,\cot^2\frac{n\pi}{2n+1}$$

are the roots of the equation

$$\sum_{r=0}^{n}(-1)^r\binom{2n+1}{2r+1}x^{n-r} = 0$$

(and we can therefore evaluate their sum by Vieta's formulae). Thus, in our particular case,

$$\cot^2\frac{\pi}{7},\ \cot^2\frac{2\pi}{7},\ \cot^2\frac{3\pi}{7}$$

are the roots of the equation

$$\sum_{r=0}^{3}(-1)^r\binom{7}{2r+1}x^{3-r} = 7x^3 - 35x^2 + 21x - 1 = 0$$

which proves once again that their sum is 5.

Example 2.20. *Show that*

$$\tan\frac{\pi}{7}\tan\frac{2\pi}{7}\tan\frac{3\pi}{7} = \sqrt{7}.$$

Solution. We have

$$\tan 7t = \frac{7\tan t - 35\tan^3 t + 21\tan^5 t - \tan^7 t}{1 - 21\tan^2 t + 35\tan^4 t - 7\tan^6 t}$$

(why? again, see the next chapter for a clever way of proving this), which yields (by replacing t with $\pi/7$, $2\pi/7$, $3\pi/7$, respectively, and by using the fact that the tangents of these numbers are not zero)

$$7 - 35x + 21x^2 - x^3 = 0 \Leftrightarrow x^3 - 21x^2 + 35x - 7 = 0$$

for $x \in \left\{\tan^2\frac{\pi}{7}, \tan^2\frac{2\pi}{7}, \tan^2\frac{3\pi}{7}\right\}$.
As the numbers from this set are mutually distinct, they must be all the solutions of the above equation, therefore their product is 7 (by Vieta's relations):

$$\tan^2\frac{\pi}{7}\tan^2\frac{2\pi}{7}\tan^2\frac{3\pi}{7} = 7.$$

Since all the tangents involved are positive, the conclusion follows.

Example 2.21. *Show that*

$$4\sin\frac{2\pi}{7} - \tan\frac{\pi}{7} = \sqrt{7}.$$

Solution. Again, by

$$\tan 7t = \frac{7\tan t - 35\tan^3 t + 21\tan^5 t - \tan^7 t}{1 - 21\tan^2 t + 35\tan^4 t - 7\tan^6 t}$$

we find that the six roots of the equation

$$x^6 - 21x^4 + 35x^2 - 7 = 0$$

are $\pm\tan\pi/7$, $\pm\tan 2\pi/7$, $\pm\tan 3\pi/7$.
Now, since

$$\begin{aligned} x^6 - 21x^4 + 35x^2 - 7 &= (x^3 - 7x)^2 - 7(x^2+1)^2 \\ &= \left(x^3 + \sqrt{7}x^2 - 7x + \sqrt{7}\right)\left(x^3 - \sqrt{7}x^2 - 7x - \sqrt{7}\right), \end{aligned}$$

we have that
$$u = \tan\frac{\pi}{7}$$
is a root either of
$$P(x) = x^3 + \sqrt{7}x^2 - 7x + \sqrt{7},$$
or of
$$Q(x) = x^3 - \sqrt{7}x^2 - 7x - \sqrt{7}.$$
Because
$$P(0)P(1) = \sqrt{7}\left(2\sqrt{7} - 6\right) < 0,$$
P has a root in the interval $(0,1)$, and that can only be $\tan\pi/7$ (from the numbers $\pm\tan\pi/7$, $\pm\tan 2\pi/7$, $\pm\tan 3\pi/7$ this is the only one between 0 and 1). Thus we have
$$u^3 + \sqrt{7}u^2 - 7u + \sqrt{7} = 0$$
which can be read as
$$\frac{8u}{1+u^2} - u = \sqrt{7},$$
meaning that
$$4\sin\frac{2\pi}{7} - \tan\frac{\pi}{7} = \sqrt{7}$$
if we use
$$\sin 2a = \frac{2\tan a}{1+\tan^2 a}.$$

Example 2.22. *Show that*
$$\sqrt[3]{\cos\frac{2\pi}{7}} + \sqrt[3]{\cos\frac{4\pi}{7}} + \sqrt[3]{\cos\frac{8\pi}{7}} = \sqrt[3]{\frac{1}{2}\left(5 - 3\sqrt[3]{7}\right)}.$$

Solution. This is a famous identity of Ramanujan. We have
$$\cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{8\pi}{7} = \cos\frac{2\pi}{7} - \cos\frac{3\pi}{7} - \cos\frac{\pi}{7} = -\frac{1}{2}$$
as we showed in Example 2.20. Then
$$\cos\frac{2\pi}{7}\cos\frac{4\pi}{7} + \cos\frac{2\pi}{7}\cos\frac{8\pi}{7} + \cos\frac{4\pi}{7}\cos\frac{8\pi}{7}$$

$$= \frac{1}{2}\left(\cos\frac{6\pi}{7} + \cos\frac{2\pi}{7} + \cos\frac{10\pi}{7} + \cos\frac{6\pi}{7} + \cos\frac{12\pi}{7} + \cos\frac{4\pi}{7}\right)$$
$$= \frac{1}{2}\left(-\cos\frac{\pi}{7} + \cos\frac{2\pi}{7} - \cos\frac{3\pi}{7} - \cos\frac{\pi}{7} + \cos\frac{2\pi}{7} - \cos\frac{3\pi}{7}\right)$$
$$= -\cos\frac{\pi}{7} + \cos\frac{2\pi}{7} - \cos\frac{3\pi}{7} = -\frac{1}{2}$$

and

$$\cos\frac{2\pi}{7}\cos\frac{4\pi}{7}\cos\frac{8\pi}{7} = \frac{\sin\frac{16\pi}{7}}{8\sin\frac{2\pi}{7}} = \frac{1}{8}.$$

Thus, letting

$$a = \sqrt[3]{2\cos\frac{2\pi}{7}},\ b = \sqrt[3]{2\cos\frac{4\pi}{7}},\ \text{and } c = \sqrt[3]{2\cos\frac{8\pi}{7}}$$

we have for a^3, b^3, and c^3 the equations

$$a^3 + b^3 + c^3 = -1,\ a^3b^3 + b^3c^3 + c^3a^3 = -2,\ a^3b^3c^3 = 1 \Rightarrow abc = 1.$$

Let $x = a + b + c$ and $y = ab + bc + ca$. Then

$$a^3 + b^3 + c^3 - 3abc = (a+b+c)(a^2 + b^2 + c^2 - ab - bc - ca)$$
$$\Rightarrow x(x^2 - 3y) = -4.$$

In a similar way (with ab, bc, ca instead),

$$-5 = (ab+bc+ca)((ab+bc+ca)^2 - 3abc(a+b+c)) = y(y^2 - 3x).$$

To solve the system for x, use the first equation to get

$$y = \frac{x^3+4}{3x}$$

and substitute. We get

$$-5 = \frac{(x^3+4)((x^3+4)^2 - 27x^3)}{27x^3}$$

$$\begin{aligned} 0 &= 135x^3 + (x^3+4)(x^6 - 19x^3 + 16) \\ &= x^9 - 15x^6 + 75x^3 + 64 = (x^3-5)^3 + 189 \\ &\Rightarrow (5-x^3)^3 = 7 \cdot 3^3 \Rightarrow x = \sqrt[3]{5 - 3\sqrt[3]{7}} \end{aligned}$$

as desired.

Example 2.23. *Prove that*

$$\sin\frac{\pi}{13} + \sin\frac{3\pi}{13} + \sin\frac{4\pi}{13} = \frac{1}{2} \cdot \sqrt{\frac{13+3\sqrt{13}}{2}}.$$

Solution. After squaring both sides (which are positive) and after transforming products into sums we need to prove that

$$\cos\frac{2\pi}{13} + \cos\frac{6\pi}{13} + \cos\frac{8\pi}{13} - 2\left(\cos\frac{4\pi}{13} + \cos\frac{10\pi}{13} + \cos\frac{12\pi}{13}\right) = \frac{1+3\sqrt{13}}{4}.$$

Let

$$\cos\frac{2\pi}{13} + \cos\frac{6\pi}{13} + \cos\frac{8\pi}{13} = x$$

and

$$\cos\frac{4\pi}{13} + \cos\frac{10\pi}{13} + \cos\frac{12\pi}{13} = y.$$

Then we compute that

$$x + y = -\frac{1}{2} \text{ and } xy = -\frac{3}{4}.$$

Indeed, the first of these is a special case of the next problem, since we have

$$\begin{aligned} x + y &= \sum_{k=1}^{6} \cos\frac{2k\pi}{13} = \frac{\sin\frac{6\pi}{13}\cos\frac{7\pi}{13}}{\sin\frac{\pi}{13}} \\ &= -\frac{\sin\frac{6\pi}{13}\cos\frac{6\pi}{13}}{\sin\frac{\pi}{13}} = -\frac{1}{2} \cdot \frac{\sin\frac{12\pi}{13}}{\sin\frac{\pi}{13}} = -\frac{1}{2}. \end{aligned}$$

The second follows by expanding and repeatedly using the formula $2\cos a\cos b = \cos(a+b) + \cos(a-b)$. We get

$$\begin{aligned} 2xy &= \cos\frac{6\pi}{13} + \cos\frac{2\pi}{13} + \cos\frac{12\pi}{13} + \cos\frac{8\pi}{13} + \cos\frac{14\pi}{13} + \cos\frac{10\pi}{13} \\ &+ \cos\frac{10\pi}{13} + \cos\frac{2\pi}{13} + \cos\frac{16\pi}{13} + \cos\frac{4\pi}{13} + \cos\frac{18\pi}{13} + \cos\frac{6\pi}{13} \\ &+ \cos\frac{12\pi}{13} + \cos\frac{4\pi}{13} + \cos\frac{18\pi}{13} + \cos\frac{2\pi}{13} + \cos\frac{20\pi}{13} + \cos\frac{4\pi}{13} \\ &= 3\left(\cos\frac{2\pi}{13} + \cos\frac{4\pi}{13} + \cos\frac{6\pi}{13} + \cos\frac{8\pi}{13} + \cos\frac{10\pi}{13} + \cos\frac{12\pi}{13}\right) \end{aligned}$$

(where we also used $\cos(2\pi - a) = \cos a$), hence

$$xy = \frac{3}{2}(x+y) = -\frac{3}{4}.$$

Thus (as x is positive and y is negative),

$$x = \frac{\sqrt{13}-1}{4} \text{ and } y = -\frac{\sqrt{13}+1}{4}.$$

Consequently,

$$\begin{aligned} &\cos\frac{2\pi}{13} + \cos\frac{6\pi}{13} + \cos\frac{8\pi}{13} - 2\left(\cos\frac{4\pi}{13} + \cos\frac{10\pi}{13} + \cos\frac{12\pi}{13}\right) \\ &= \frac{\sqrt{13}-1}{4} + \frac{\sqrt{13}+1}{2} = \frac{1+3\sqrt{13}}{4}. \end{aligned}$$

Example 2.24. *Evaluate*

$$\sum_{k=1}^{n} \cos kx.$$

Solution. Assuming that $x \neq 2m\pi$, m an integer, we multiply by $2\sin x/2$. From the product-to-sum formula we get

$$\begin{aligned} 2\sum_{k=1}^{n} \sin\frac{x}{2}\cos kx &= \sum_{k=1}^{n}\left(\sin\left(k+\frac{1}{2}\right)x - \sin\left(k-\frac{1}{2}\right)x\right) \\ &= \sin\left(n+\frac{1}{2}\right)x - \sin\frac{x}{2} = 2\sin\frac{n\pi}{2}\cos\frac{(n+1)\pi}{2}. \end{aligned}$$

Therefore we get

$$\sum_{k=1}^{n} \cos kx = \frac{\sin\left(n+\frac{1}{2}\right)x}{2\sin\frac{x}{2}} - \frac{1}{2} = \frac{\sin\frac{nx}{2}\cos\frac{(n+1)x}{2}}{\sin\frac{x}{2}}.$$

Clearly, when $x = 2m\pi$, m is an integer, the answer is n.

Example 2.25. *Prove that, for all real numbers $x \neq 2m\pi$ (with integer m), we have*

$$\sin x + \sin 2x + \sin 3x + \cdots + \sin nx = \frac{\sin\frac{nx}{2}\sin\frac{(n+1)x}{2}}{\sin\frac{x}{2}}.$$

Solution. As in the previous problem, we multiply the sum with $2\sin x/2$, and use formulae for transforming products into sums and vice-versa:

$$\begin{aligned} 2\sin\frac{x}{2}\sum_{k=1}^{n}\sin kx &= \sum_{k=1}^{n} 2\sin\frac{x}{2}\sin kx = \sum_{k=1}^{n}\left(\cos\left(k-\frac{1}{2}\right)x - \cos\left(k+\frac{1}{2}\right)x\right) \\ &= \cos\frac{x}{2} - \cos\frac{(2n+1)x}{2} = 2\sin\frac{nx}{2}\sin\frac{(n+1)x}{2}, \end{aligned}$$

and this is what we had to prove. Of course, for $x = 2m\pi$ the sum is 0. Note that mathematical induction can be used in such problems, too.
Also, observe (and prove) that the slightly more general identity

$$\sin x + \sin(x+a) + \cdots + \sin(x+(n-1)a) = \frac{\sin\frac{na}{2}\sin\left(x+\frac{n-1}{2}a\right)}{\sin\frac{a}{2}}$$

holds for any real numbers x and $a \neq 2m\pi$, $m \in \mathbb{Z}$.

Example 2.26. *Prove that*

$$\cos x + \cos 3x + \cos 5x + \cdots + \cos(2n-1)x = \frac{\sin 2nx}{2\sin x},$$

where $x \neq m\pi$, $m \in \mathbb{Z}$.

Solution. Indeed we have

$$\begin{aligned}2\sin x\sum_{k=1}^{n}\cos(2k-1)x &= \sum_{k=1}^{n}(\sin 2kx-\sin(2k-2)x)\\ &= \sin 2nx.\end{aligned}$$

Actually, the more general identity

$$\cos x+\cos(x+a)+\cdots+\cos(x+(n-1)a)=\frac{\sin\dfrac{na}{2}\cos\left(x+\dfrac{n-1}{2}a\right)}{\sin\dfrac{a}{2}}$$

holds for $a\neq 2m\pi$. Can you prove it by inducting on n, or by multiplying the sum with $\sin a/2$?

Example 2.27. *Prove that, for $x\neq m\pi$ with integer m,*

$$\sin x+2\sin 2x+3\sin 3x+\cdots+n\sin nx=\frac{(n+1)\sin nx-n\sin(n+1)x}{4\sin^2\dfrac{x}{2}}.$$

Solution. We differentiate with respect to x the equation (that we obtained earlier)

$$\sum_{k=1}^{n}\cos kx=\frac{\sin\dfrac{(2n+1)x}{2}}{2\sin\dfrac{x}{2}}-\frac{1}{2}$$

and thus get

$$\begin{aligned}-\sum_{k=1}^{n}k\sin kx &= \frac{\dfrac{2n+1}{2}\sin\dfrac{x}{2}\cos\dfrac{(2n+1)x}{2}-\dfrac{1}{2}\sin\dfrac{(2n+1)x}{2}\cos\dfrac{x}{2}}{2\sin^2\dfrac{x}{2}}\\ &= \frac{(2n+1)(\sin(n+1)x-\sin nx)-(\sin(n+1)x+\sin nx)}{8\sin^2\dfrac{x}{2}}\\ &= \frac{2n\sin(n+1)x-(2n+2)\sin nx}{8\sin^2\dfrac{x}{2}},\end{aligned}$$

and the desired result follows.
We can also evaluate the sum multiplied by $2\sin x/2$:

$$2\sin\frac{x}{2}\sum_{k=1}^{n} k\sin kx = \sum_{k=1}^{n} k\left(\cos\frac{(2k-1)x}{2} - \cos\frac{(2k+1)x}{2}\right)$$
$$= \sum_{k=1}^{n}\cos\frac{(2k-1)x}{2} - n\cos\frac{(2n+1)x}{2}.$$

Now, by the previous example, we have

$$\sum_{k=1}^{n}\cos\frac{(2k-1)x}{2} = \frac{\sin nx}{2\sin\frac{x}{2}}$$

and the result follows after a few more manipulations with product-to-sum formulae (actually only one such formula is needed), and, of course, after dividing by $2\sin x/2$. The method of induction is also available in such an exercise: try it! Also, try to prove (with or without mathematical induction) that

$$\cos x + 2\cos 2x + \cdots + n\cos nx = \frac{(n+1)\cos nx - n\cos(n+1)x - 1}{4\sin^2\frac{x}{2}}$$

holds – again for any real x different from any even multiple of π.

Example 2.28. (USAMO, 1992) *Prove that*

$$\frac{1}{\cos 0^\circ\cos 1^\circ} + \frac{1}{\cos 1^\circ\cos 2^\circ} + \cdots + \frac{1}{\cos 88^\circ\cos 89^\circ} = \frac{\cos 1^\circ}{\sin^2 1^\circ}.$$

Solution. Multiplying the relation by $\sin 1^\circ$, we obtain

$$\frac{\sin 1^\circ}{\cos 0^\circ\cos 1^\circ} + \frac{\sin 1^\circ}{\cos 1^\circ\cos 2^\circ} + \cdots + \frac{\sin 1^\circ}{\cos 88^\circ\cos 89^\circ} = \frac{\cos 1^\circ}{\sin 1^\circ}.$$

This can be rewritten as

$$\frac{\sin(1^\circ - 0^\circ)}{\cos 0^\circ\cos 1^\circ} + \frac{\sin(2^\circ - 1^\circ)}{\cos 1^\circ\cos 2^\circ} + \cdots + \frac{\sin(89^\circ - 88^\circ)}{\cos 88^\circ\cos 89^\circ} = \cot 1^\circ.$$

From the identity

$$\tan a - \tan b = \frac{\sin(a-b)}{\cos a \cos b},$$

it follows that the left side equals

$$\sum_{k=1}^{89} [\tan k^\circ - \tan(k-1)^\circ] = \tan 89^\circ - \tan 0^\circ = \cot 1^\circ,$$

and the identity is proved.

Example 2.29. (IMO Longlist 1966) *Prove that for every natural number n, and for every real number $x \neq \frac{k\pi}{2^l}$, $l = 0, 1, \ldots, n$, k any integer*

$$\frac{1}{\sin 2x} + \frac{1}{\sin 4x} + \cdots + \frac{1}{\sin 2^n x} = \cot x - \cot 2^n x.$$

Solution. By the double angle formula for cosine,

$$\cos 2^k x = 2\cos^2 2^{k-1} x - 1.$$

Dividing both sides by $\sin 2^k x = 2 \sin 2^{k-1} x \cos 2^{k-1} x$ (which is nonzero for the given values of x),

$$\cot 2^k x = \cot 2^{k-1} x - \frac{1}{\sin 2^k x} \Rightarrow \frac{1}{\sin 2^k x} = \cot 2^{k-1} x - \cot 2^k x.$$

Hence,

$$\sum_{k=1}^{n} \frac{1}{\sin 2^k x} = \sum_{k=1}^{n} (\cot 2^{k-1} x - \cot 2^k x) = \cot x - \cot 2^n x,$$

and the conclusion follows.

Example 2.30. (USAMO 1996) *Prove that the average of the numbers $n \sin n^\circ$, $n = 2, 4, 6, \ldots, 180$ is $\cot 1^\circ$.*

Solution. Instead of showing that

$$\frac{2\sin 2^\circ + 4\sin 4^\circ + 6\sin 6^\circ + \cdots + 180\sin 180^\circ}{90} = \cot 1^\circ,$$

we will prove the equivalent form

$$2\sin 1^\circ \sin 2^\circ + 4\sin 1^\circ \sin 4^\circ + 6\sin 1^\circ \sin 6^\circ + \cdots + 180\sin 1^\circ \sin 180^\circ = 90\cos 1^\circ.$$

By the product-to-sum formulas, the expression simplifies to

$$\begin{aligned}\sum_{n=1}^{90} 2n\sin 1^\circ \sin 2n^\circ &= \sum_{n=1}^{90} n[\cos(2n-1)^\circ - \cos(2n+1)^\circ] \\ &= -90\cos 181^\circ + \sum_{n=1}^{90}\cos(2n-1)^\circ \\ &= 90\cos 1^\circ + \sum_{n=1}^{90}\cos(2n-1)^\circ.\end{aligned}$$

It now suffices to show that

$$\sum_{n=1}^{90}\cos(2n-1)^\circ = 0.$$

Using the identity $\cos n^\circ = -\cos(180-n)^\circ$,

$$\begin{aligned}\sum_{n=1}^{90}\cos(2n-1)^\circ &= \cos 1^\circ + \cos 3^\circ + \cdots + \cos 89^\circ + \cos 91^\circ + \cdots + \cos 177^\circ + 179^\circ \\ &= \cos 1^\circ + \cos 3^\circ + \cdots + \cos 89^\circ - (\cos 89^\circ + \cdots + \cos 3^\circ + \cos 1^\circ) \\ &= 0.\end{aligned}$$

Example 2.31. *Evaluate the product*

$$\prod_{k=1}^{n}\left(1 - \tan^2\frac{2^k\pi}{2^n+1}\right).$$

Solution. Since

$$1-\tan^2\theta = \frac{\cos 2\theta}{\cos^2\theta},$$

we can rewrite the product as

$$\prod_{k=1}^{n}\left(1-\tan^2\frac{2^k\pi}{2^n+1}\right) = \prod_{k=1}^{n}\frac{\cos\dfrac{2^{k+1}\pi}{2^n+1}}{\cos^2\dfrac{2^k\pi}{2^n+1}}$$

which is equal to

$$\left(\frac{\cos\dfrac{4\pi}{2^n+1}}{\cos^2\dfrac{2\pi}{2^n+1}}\right)\left(\frac{\cos\dfrac{8\pi}{2^n+1}}{\cos^2\dfrac{4\pi}{2^n+1}}\right)\left(\frac{\cos\dfrac{16\pi}{2^n+1}}{\cos^2\dfrac{8\pi}{2^n+1}}\right)\cdots\left(\frac{\cos\dfrac{2^{n+1}\pi}{2^n+1}}{\cos^2\dfrac{2^n\pi}{2^n+1}}\right).$$

We can cross cancel some of the factors from the numerator to get

$$\frac{\cos\dfrac{2^{n+1}\pi}{2^n+1}}{\cos^2\dfrac{2\pi}{2^n+1}\cos\dfrac{4\pi}{2^n+1}\cos\dfrac{8\pi}{2^n+1}\cdots\cos\dfrac{2^n\pi}{2^n+1}}$$

$$=\frac{1}{\cos\dfrac{2\pi}{2^n+1}\cos\dfrac{4\pi}{2^n+1}\cos\dfrac{8\pi}{2^n+1}\cdots\cos\dfrac{2^n\pi}{2^n+1}}$$

because

$$\cos\frac{2^{n+1}\pi}{2^n+1} = \cos\left(2\pi-\frac{2^{n+1}\pi}{2^n+1}\right) = \cos\frac{2\pi}{2^n+1}.$$

Since the double angle formula for the sine gives the general formula

$$\cos x\cos 2x\cdots\cos 2^{n-1}x = \frac{\sin 2x}{2\sin x}\cdot\frac{\sin 4x}{2\sin 2x}\cdots\frac{\sin 2^n x}{2\sin 2^{n-1}x} = \frac{\sin 2^n x}{2^n\sin x},$$

the product telescopes to

$$\frac{2^n\sin\dfrac{2\pi}{2^n+1}}{\sin\dfrac{2^{n+1}\pi}{2^n+1}} = \frac{2^n\sin\dfrac{2\pi}{2^n+1}}{-\sin\dfrac{2\pi}{2^n+1}} = -2^n.$$

Example 2.32. (IMO Longlist 1967) *Prove the identity*

$$\sum_{k=0}^{n}\binom{n}{k}\left(\tan\frac{x}{2}\right)^{2k}\left(1+\frac{2^k}{\left(1-\tan^2\frac{x}{2}\right)^k}\right)=\sec^{2n}\frac{x}{2}+\sec^n x$$

for any natural number n and any angle x.

Solution. This problem relies on binomial coefficients and the binomial theorem, the reader unfamiliar with these should consult the *Introduction*, or the chapter *Combinatorial Identities and Generating Functions.* For the solution, denote

$$A=\tan^2\frac{x}{2}.$$

Then

$$A+1=\tan^2\frac{x}{2}+1=\sec^2\frac{x}{2}$$

and by the half-angle formula

$$\frac{1+A}{1-A}=\frac{1+\tan^2\frac{x}{2}}{1-\tan^2\frac{x}{2}}=\frac{1}{\cos x}=\sec x.$$

Thus

$$\sum_{k=0}^{n}\binom{n}{k}\left(\tan\frac{x}{2}\right)^{2k}\left(1+\frac{2^k}{\left(1-\tan^2\frac{x}{2}\right)^k}\right)$$

$$=\sum_{k=0}^{n}\binom{n}{k}\left(\tan^2\frac{x}{2}\right)^{k}\left(1+\frac{2^k}{\left(1-\tan^2\frac{x}{2}\right)^k}\right)$$

$$=\sum_{k=0}^{n}\binom{n}{k}A^k\left(1+\frac{2^k}{(1-A)^k}\right)=\sum_{k=0}^{n}\binom{n}{k}A^k+\sum_{k=0}^{n}\binom{n}{k}A^k\cdot\frac{2^k}{(1-A)^k}$$

$$=\sum_{k=0}^{n}\binom{n}{k}A^k+\sum_{k=0}^{n}\binom{n}{k}\left(\frac{2A}{1-A}\right)^k=(A+1)^n+\left(\frac{2A}{1-A}+1\right)^n$$

$$= (A+1)^n + \left(\frac{1+A}{1-A}\right)^n = \left(\sec^2\frac{x}{2}\right)^n + (\sec x)^n = \sec^{2n}\frac{x}{2} + \sec^n x,$$

and the problem is solved.

Example 2.33. *Suppose $x \neq n\pi$. Let*

$$S_n = \frac{4^0}{\cos^2 x} + \frac{4^1}{\cos^2 2x} + \frac{4^2}{\cos^2 4x} + \cdots + \frac{4^n}{\cos^2 2^n x}.$$

(a) Prove that

$$\frac{1}{\cos^2 x} = \frac{4}{\sin^2 2x} - \frac{1}{\sin^2 x}.$$

(b) Prove that

$$S_n = \frac{4^{n+1}}{\sin^2 2^{n+1}x} - \frac{1}{\sin^2 x}.$$

Solution. (a) We have, indeed,

$$\frac{4}{\sin^2 2x} - \frac{1}{\sin^2 x} = \frac{1}{\sin^2 x\cos^2 x} - \frac{1}{\sin^2 x} = \frac{1-\cos^2 x}{\sin^2 x\cos^2 x} = \frac{1}{\cos^2 x}.$$

(b) According to the first part, we have

$$\sum_{k=0}^{n}\frac{4^k}{\cos^2 2^k x} = \sum_{k=0}^{n}\left(\frac{4^{k+1}}{\sin^2 2^{k+1}x} - \frac{4^k}{\sin^2 2^k x}\right) = \frac{4^{n+1}}{\sin^2 2^{n+1}x} - \frac{1}{\sin^2 x}.$$

Example 2.34. *Prove that*

$$\prod_{n=1}^{89}\ln\tan\frac{n\pi}{180} = \sum_{n=1}^{89}\ln\tan\frac{n\pi}{180}.$$

Solution. Notice that $\prod_{n=1}^{89}\ln\tan\frac{n\pi}{180}$ has the term $\ln\tan\frac{45\pi}{180} = 0$, so

$$\prod_{n=1}^{89}\ln\tan\frac{n\pi}{180} = 0.$$

As for the sum, we have

$$\begin{aligned}
\sum_{n=1}^{89} \ln \tan \frac{n\pi}{180} &= \ln \prod_{n=1}^{89} \tan \frac{n\pi}{180} \\
&= \ln \left(\prod_{n=1}^{44} \tan \frac{n\pi}{180} \cdot \prod_{n=46}^{90} \tan \frac{n\pi}{180} \cdot \tan \frac{45\pi}{180} \right) \\
&= \ln \left(\prod_{n=1}^{44} \left(\tan \frac{n\pi}{180} \cdot \tan \frac{(90-n)\pi}{180} \right) \cdot \tan \frac{45\pi}{180} \right) \\
&= \ln \left(\prod_{n=1}^{44} \left(\tan \frac{n\pi}{180} \cdot \cot \frac{n\pi}{180} \right) \cdot \tan \frac{45\pi}{180} \right) \\
&= \ln \left(\prod_{n=1}^{44} 1 \cdot 1 \right) \\
&= \ln 1 = 0.
\end{aligned}$$

Example 2.35. *Find in closed form*

$$S = \sin x \sin 3x + \sin \frac{x}{2} \sin \frac{3x}{2} + \sin \frac{x}{2^2} \sin \frac{3x}{2^2} + \cdots + \sin \frac{x}{2^{n-1}} \sin \frac{3x}{2^{n-1}}.$$

Solution. We have

$$\begin{aligned}
2S &= 2 \sin x \sin 3x + 2 \sin \frac{x}{2} \sin \frac{3x}{2} + 2 \sin \frac{x}{2^2} \sin \frac{3x}{2^2} + \cdots + 2 \sin \frac{x}{2^{n-1}} \sin \frac{3x}{2^{n-1}} \\
&= \cos 2x - \cos 4x + \cos x - \cos 2x + \cos \frac{x}{2} - \cos x + \cos \frac{x}{4} - \cos \frac{x}{2} \\
&+ \cdots + \cos \frac{x}{2^{n-2}} - \cos \frac{x}{2^{n-3}},
\end{aligned}$$

hence, after cancellations,

$$S = \frac{1}{2} \left(-\cos 4x + \cos \frac{x}{2^{n-2}} \right).$$

Example 2.36. *Find in closed form*

$$S = \cos \frac{x}{2} + 2 \cos \frac{x}{2} \cos \frac{x}{2^2} + \cdots + 2^{n-1} \cos \frac{x}{2} \cos \frac{x}{2^2} \cdots \cos \frac{x}{2^n}.$$

Solution. As we have seen before, there is a standard evaluation of the product that represents the general term of the sum, by using the formula for the sine of the double angle. Namely we have

$$\prod_{j=1}^{k} \cos\frac{x}{2^j} = \prod_{j=1}^{k} \frac{\sin\dfrac{x}{2^{j-1}}}{2\sin\dfrac{x}{2^j}} = \frac{\sin x}{2^k \sin\dfrac{x}{2^k}}.$$

Using the formula for the difference of cotangents, that is,

$$\cot a - \cot b = \frac{\sin(b-a)}{\sin a \sin b},$$

we can write

$$\frac{1}{\sin\dfrac{x}{2^k}} = \frac{\sin\left(\dfrac{x}{2^k} - \dfrac{x}{2^{k+1}}\right)}{\sin\dfrac{x}{2^k}\sin\dfrac{x}{2^{k+1}}} = \cot\frac{x}{2^{k+1}} - \cot\frac{x}{2^k}.$$

Thus,

$$\begin{aligned}\sum_{k=1}^{n} 2^{k-1} \prod_{j=1}^{k} \cos\frac{x}{2^j} &= \frac{1}{2}\sin x \sum_{k=1}^{n}\left(\cot\frac{x}{2^{k+1}} - \cot\frac{x}{2^k}\right)\\ &= \frac{1}{2}\sin x\left(\cot\frac{x}{2^{n+1}} - \cot\frac{x}{2}\right).\end{aligned}$$

Example 2.37. *Prove that*

$$\frac{1}{2}\tan\frac{x}{2} + \frac{1}{2^2}\tan\frac{x}{2^2} + \cdots + \frac{1}{2^n}\tan\frac{x}{2^n} = \frac{1}{2^n}\cot\frac{x}{2^n} - \cot x,$$

for $x \neq k\pi$, with $k \in \mathbb{Z}$.

Solution. By using the formula (which we invite the reader to prove)

$$\tan a = \cot a - 2\cot 2a$$

the sum telescopes yielding the desired result:

$$\sum_{k=1}^{n} \frac{1}{2^k}\tan\frac{x}{2^k} = \sum_{k=1}^{n}\left(\frac{1}{2^k}\cot\frac{x}{2^k} - \frac{1}{2^{k-1}}\cot\frac{x}{2^{k-1}}\right) = \frac{1}{2^n}\cot\frac{x}{2^n} - \cot x.$$

Example 2.38. *Prove that*

$$\sum_{k=1}^{n} \tan kx \tan(k+1)x = -(n+1) + \frac{\tan(n+1)x}{\tan x}$$

holds for any real number $x \neq m\pi/2$, with integer m.

Solution. We have

$$\tan x = \tan((k+1)x - kx) = \frac{\tan(k+1)x - \tan kx}{1 + \tan kx \tan(k+1)x},$$

implying

$$\tan kx \tan(k+1)x = -1 + \frac{1}{\tan x}(\tan(k+1)x - \tan kx).$$

Consequently,

$$\begin{aligned}\sum_{k=1}^{n} \tan kx \tan(k+1)x &= \sum_{k=1}^{n} \left(-1 + \frac{1}{\tan x}(\tan(k+1)x - \tan kx)\right) \\ &= -n + \frac{1}{\tan x}(\tan(n+1)x - \tan x) \\ &= -(n+1) + \frac{\tan(n+1)x}{\tan x},\end{aligned}$$

as we intended to obtain.

Example 2.39. *Prove that*

$$1 + \frac{\cos x}{\cos^1 x} + \frac{\cos(2x)}{\cos^2 x} + \cdots + \frac{\cos(nx)}{\cos^n x} = \frac{\sin(n+1)x}{\sin x \cos^n x}$$

for any real number $x \neq m\pi/2$, $m \in \mathbb{Z}$.

Solution. Careful examination of the given result strongly suggests the telescoping formula

$$\frac{\cos kx}{\cos^k x} = \frac{\sin(k+1)x}{\sin x \cos^k x} - \frac{\sin kx}{\sin x \cos^{k-1} x}$$

for $k = 0, 1, 2, \ldots$. Indeed

$$\frac{\sin(k+1)x}{\sin x \cos^k x} - \frac{\sin kx}{\sin x \cos^{k-1} x} = \frac{\sin(k+1)x - \sin kx \cos x}{\sin x \cos^k x}$$
$$= \frac{\sin x \cos kx}{\sin x \cos^k x} = \frac{\cos kx}{\cos^k x}.$$

Now the evaluation of the sum is clear:

$$\sum_{k=0}^{n} \frac{\cos kx}{\cos^k x} = \sum_{k=0}^{n} \left(\frac{\sin(k+1)x}{\sin x \cos^k x} - \frac{\sin kx}{\sin x \cos^{k-1} x} \right)$$
$$= \frac{\sin(n+1)x}{\sin x \cos^n x},$$

finishing the solution.

Example 2.40. *Let n be an be an integer with $n \geq 2$. Prove that*

$$\prod_{k=1}^{n} \tan \left[\frac{\pi}{3} \left(1 + \frac{3^k}{3^n - 1} \right) \right] = \prod_{k=1}^{n} \cot \left[\frac{\pi}{3} \left(1 - \frac{3^k}{3^n - 1} \right) \right].$$

Solution. Let

$$a_k = \tan \frac{3^{k-1}\pi}{3^n - 1}.$$

It is easy to see that, for all $n > 1$ and for all $k \in \mathbb{N}$, a_k is defined and different from any of the numbers $\pm\sqrt{3}$ and $\pm 1/\sqrt{3}$.
Our equation can be written in the following equivalent forms

$$\prod_{k=1}^{n} \frac{\sqrt{3} + a_k}{1 - \sqrt{3}a_k} = \prod_{k=1}^{n} \frac{1 + \sqrt{3}a_k}{\sqrt{3} - a_k}$$
$$\Leftrightarrow \prod_{k=1}^{n} \frac{3 - a_k^2}{1 - 3a_k^2} = 1$$
$$\Leftrightarrow \prod_{k=1}^{n} \frac{1}{a_k} \cdot \frac{3a_k - a_k^3}{1 - 3a_k^2} = 1$$
$$\Leftrightarrow \prod_{k=1}^{n} \frac{a_{k+1}}{a_k} = 1$$
$$\Leftrightarrow a_{n+1} = a_1.$$

The last one is trivially true since

$$\frac{3^n\pi}{3^n-1}=\pi+\frac{\pi}{3^n-1}.$$

Note that we used again the formula for the tangent of the triple angle, when we replaced $\dfrac{3a_k-a_k^3}{1-3a_k^2}$ by a_{k+1}.

Example 2.41. *Prove that*

$$\sum_{k=1}^{n}(-1)^{k-1}\cos\frac{k\pi}{2n+1}=\frac{1}{2}.$$

Solution. Remember the formula

$$\sum_{k=1}^{n}\cos(x+(k-1)a)=\frac{\sin\dfrac{na}{2}\cos\left(x+\dfrac{(n-1)a}{2}\right)}{\sin\dfrac{a}{2}}.$$

After using $(-1)^j\cos t=\cos(t+j\pi)$ for every term of the sum, we can apply this formula here with

$$x=\frac{\pi}{2n+1}\quad\text{and}\quad a=\frac{2(n+1)\pi}{2n+1}.$$

We will have

$$\begin{aligned}\sum_{k=1}^{n}(-1)^{k-1}\cos\frac{k\pi}{2n+1}&=\sum_{k=1}^{n}\cos\left(\frac{k\pi}{2n+1}+(k-1)\pi\right)\\&=\frac{\sin\dfrac{n(n+1)\pi}{2n+1}\cos\dfrac{n^2\pi}{2n+1}}{\sin\dfrac{(n+1)\pi}{2n+1}}\end{aligned}$$

$$= \frac{\sin\dfrac{n(2n+1)\pi}{2n+1} + \sin\dfrac{n\pi}{2n+1}}{2\sin\dfrac{(n+1)\pi}{2n+1}} = \frac{\sin\dfrac{n\pi}{2n+1}}{2\sin\dfrac{(n+1)\pi}{2n+1}} = \frac{1}{2}$$

because $\sin n\pi = 0$ and $\dfrac{n\pi}{2n+1} + \dfrac{(n+1)\pi}{2n+1} = \pi$.

Example 2.42. *Prove that for all $n \geq 1$*

$$\prod_{k=1}^{n} \cos\frac{k\pi}{2n+1} = \frac{1}{2^n}.$$

Solution. Since $\sin\dfrac{k\pi}{2n+1}$ is nonzero for $1 \leq k \leq n$, it suffices to prove that

$$2^n \prod_{k=1}^{n} \sin\frac{k\pi}{2n+1} \prod_{k=1}^{n} \cos\frac{k\pi}{2n+1} = \prod_{k=1}^{n} \sin\frac{k\pi}{2n+1}.$$

Using the formula $\sin 2x = 2\sin x\cos x$, the left-hand side actually is

$$\begin{aligned} 2^n \prod_{k=1}^{n} \sin\frac{k\pi}{2n+1} \prod_{k=1}^{n} \cos\frac{k\pi}{2n+1} &= \prod_{k=1}^{n} \left(2\sin\frac{k\pi}{2n+1}\cos\frac{2\pi}{2n+1}\right) \\ &= \prod_{k=1}^{n} \sin\frac{2k\pi}{2n+1}, \end{aligned}$$

hence it suffices to establish the identity

$$\prod_{k=1}^{n} \sin\frac{k\pi}{2n+1} = \prod_{k=1}^{n} \sin\frac{2k\pi}{2n+1}.$$

Since $\sin(\pi - x) = \sin x$, we can write

$$\begin{aligned} \prod_{k=1}^{n} \sin\frac{k\pi}{2n+1} &= \prod_{1\leq k\leq n,\ 2|k} \sin\frac{k\pi}{2n+1} \cdot \prod_{1\leq k\leq n,\ 2|k+1} \sin\left(\pi - \frac{k\pi}{2n+1}\right) \\ &= \prod_{1\leq k\leq n,\ 2|k} \sin\frac{k\pi}{2n+1} \cdot \prod_{1\leq k\leq n,\ 2|k+1} \sin\left(\frac{(2n+1-k)\pi}{2n+1}\right). \end{aligned}$$

When k runs over all odd numbers between 1 and n, the number $2n+1-k$ runs over all even numbers between $n+1$ and $2n$ and the result follows.

Example 2.43. *Evaluate*

$$\sum_{k=1}^{n} 2^{k-1} \frac{\cos 2^{k-1}x}{\sin^2 2^{k-1}x}.$$

Solution. We have

$$\frac{\cos t}{\sin^2 t} = \frac{2\cos^2 \frac{t}{2} - 1}{4\sin^2 \frac{t}{2} \cos^2 \frac{t}{2}} = \frac{1}{2\sin^2 \frac{t}{2}} - \frac{1}{\sin^2 t},$$

therefore

$$\begin{aligned}\sum_{k=1}^{n} 2^{k-1} \frac{\cos 2^{k-1}x}{\sin^2 2^{k-1}x} &= \sum_{k=1}^{n} \left(\frac{2^{k-2}}{\sin^2 2^{k-2}x} - \frac{2^{k-1}}{\sin^2 2^{k-1}x} \right) \\ &= \frac{1}{2\sin^2 \frac{x}{2}} - \frac{2^{n-1}}{\sin^2 2^{n-1}x}.\end{aligned}$$

Example 2.44. *Prove that for real numbers a and b, with $\sin b \neq 0$, we have*

$$\sum_{k=1}^{n} \frac{1}{\cos(a+(k-1)b)\cos(a+kb)} = \frac{1}{\sin b}(\tan(a+nb) - \tan a).$$

Solution. We have

$$\begin{aligned}\frac{1}{\cos(a+(k-1)b)\cos(a+kb)} &= \frac{1}{\sin b} \cdot \frac{\sin((a+kb)-(a+(k-1)b))}{\cos(a+(k-1)b)\cos(a+kb)} \\ &= \frac{1}{\sin b}\left(\tan(a+kb) - \tan(a+(k-1)b)\right),\end{aligned}$$

hence

$$\begin{aligned}\sum_{k=1}^{n} \frac{1}{\cos(a+(k-1)b)\cos(a+kb)} &= \sum_{k=1}^{n} (\tan(a+kb) - \tan(a+(k-1)b)) \\ &= \frac{1}{\sin b}(\tan(a+nb) - \tan a).\end{aligned}$$

Example 2.45. (Baltic Way 2014) *Show that*

$$\cos(56^\circ)\cdot\cos(2\cdot 56^\circ)\cdot\cos(2^2\cdot 56^\circ)\cdot\ldots\cdot\cos(2^{23}\cdot 56^\circ)=\frac{1}{2^{24}}.$$

Solution. Of course,

$$\sin x\prod_{k=0}^{23}\cos 2^k x=\frac{1}{2^{24}}\sin 2^{24}x.$$

For $x=56$, we have

$$2^{24}\cdot 56\equiv 56 \pmod{360},$$

since it is equivalent to

$$7(2^{24}-1)\equiv 0 \pmod{45},$$

and this follows either from Euler's theorem (using $\phi(45)=24$) or from direct computation that $5|2^4-1$ and $9|2^6-1$. Thus $\sin 2^{24}x=\sin x$ and we're done.

Example 2.46. *Compute, for a given real number x,*

$$\sum_{k=1}^{n}\sin\frac{x}{3^k}\sin\frac{2x}{3^k}.$$

Solution. Using the formula

$$2\sin a\sin b=\cos(a-b)-\cos(a+b),$$

we obtain

$$\sin\frac{x}{3^k}\sin\frac{2x}{3^k}=\frac{1}{2}\left(\cos\frac{x}{3^k}-\cos\frac{x}{3^{k-1}}\right).$$

We obtain therefore a telescopic sum, with value

$$\sum_{k=0}^{n}\sin\frac{x}{3^k}\sin\frac{2x}{3^k}=\frac{1}{2}\left(\cos\frac{x}{3^n}-\cos x\right).$$

Example 2.47. *Evaluate*

$$\sum_{k=0}^{n} \arctan \frac{1}{k^2+k+1},$$

where arctan *stands for the arctangent function.*

Solution. In the solution we will use the subtraction formula for the tangent

$$\tan(a-b) = \frac{\tan a - \tan b}{1+\tan a \tan b},$$

which gives the formula for the arctangent

$$\arctan u - \arctan v = \arctan \frac{u-v}{1+uv}.$$

For simplicity, set $a_k = \arctan k$. Then

$$\begin{aligned}\tan(a_{k+1}-a_k) &= \frac{\tan a_{k+1} - \tan a_k}{1+\tan a_{k+1}\tan a_k} \\ &= \frac{k+1-k}{1+k(k+1)} = \frac{1}{k^2+k+1.}\end{aligned}$$

Hence the sum we evaluate is equal to

$$\begin{aligned}\sum_{k=0}^{n} \arctan(\tan(a_{k+1}-a_k)) &= \sum_{k=0}^{n}(a_{k+1}-a_k) \\ &= a_{n+1}-a_0 \\ &= \arctan(n+1).\end{aligned}$$

Example 2.48. *Prove that*

$$\sum_{k=1}^{n} \operatorname{arccot}\left(2k^2\right) = \operatorname{arccot}\left(1+\frac{1}{n}\right).$$

Solution. Because $\operatorname{arccot} x = \arctan\left(\frac{1}{x}\right)$, we get

$$\begin{aligned}
\sum_{k=1}^{n} \operatorname{arccot}(2k^2) &= \sum_{k=1}^{n} \arctan\left(\frac{1}{2k^2}\right) = \sum_{k=1}^{n} \arctan \frac{(2k+1)-(2k-1)}{1+(2k-1)(2k+1)} \\
&= \sum_{k=1}^{n} (\arctan(2k+1) - \arctan(2k-1)) \\
&= \arctan(2n+1) - \arctan(1) \\
&= \arctan \frac{2n+1-1}{1+(2n+1)\cdot 1} \\
&= \arctan \frac{2n}{2n+2} = \operatorname{arccot} \frac{n+1}{n}.
\end{aligned}$$

Chapter 3

Complex Numbers and de Moivre's Formula

Every complex number z can be written in the from

$$z = x + iy = |z|(\cos u + i \sin u),$$

where $|z| = \sqrt{x^2 + y^2}$ is the modulus of z. The most famous formulas that help us in dealing with complex numbers are

- Euler's formula:
$$e^{ix} = \cos x + i \sin x.$$

- de Moivre's formula:
$$\cos nx + i \sin nx = (\cos x + i \sin x)^n.$$

Using de Moivre's Theorem, we can develop an understanding of roots of complex numbers, starting with roots of unity.
To say that $z = |z|(\cos x + i \sin x)$ is an n^{th} root of unity means that $z^n = 1$, and this requires

$$|z|^n(\cos nx + i \sin nx) = 1.$$

To satisfy the requirement, we need $|z| = 1$ and $\cos nx = 1$. The latter is satisfied if and only if $x = 2k\pi/n$, where k is an integer. It follows that every n^{th} root of unity is of the form ε^k, where

$$\varepsilon = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n} = e^{\frac{2i\pi}{n}}.$$

Let us look at some examples.

Example 3.1. *If r and t are real numbers, and $-1 < r < 1$, show that:*

(a) $\displaystyle\sum_{k=0}^{\infty} r^k \cos kt = \frac{1 - r\cos t}{1 - 2r\cos t + r^2}.$

(b) $\displaystyle\sum_{k=0}^{\infty} r^k \sin kt = \frac{r\sin t}{1 - 2r\cos t + r^2}.$

Solution. We first prove that, for any positive integer n,

$$A_n = \sum_{k=0}^{n-1} r^k \cos kt = \frac{1 - r\cos t - r^n\cos nt + r^{n+1}\cos(n-1)t}{1 - 2r\cos t + r^2}$$

and

$$B_n = \sum_{k=0}^{n-1} r^k \sin kt = \frac{r\sin t - r^n\sin nt + r^{n+1}\sin(n-1)t}{1 - 2r\cos t + r^2}.$$

Indeed, with $z = r(\cos t + i\sin t)$, we have

$$A_n + iB_n = \sum_{k=0}^{n-1} r^k(\cos kt + i\sin kt) = \sum_{k=0}^{n-1} z^k$$

$$= \frac{1 - z^n}{1 - z} = \frac{1 - r^n\cos nt - ir^n\sin nt}{1 - r\cos t - ir\sin t}$$

$$= \frac{(1 - r^n\cos nt - ir^n\sin nt)(1 - r\cos t + ir\sin t)}{(1 - r\cos t - ir\sin t)(1 - r\cos t + ir\sin t)}$$

$$= \frac{1 - r\cos t - r^n\cos nt + r^{n+1}\cos(n-1)t + i(r\sin t - r^n\sin nt + r^{n+1}\sin(n-1)t)}{1 - 2r\cos t + r^2}$$

and the conclusion follows by equating the real and imaginary parts from the left and from the right. Because $|r| < 1$, we have

$$\begin{aligned}\lim_{n\to\infty} r^n \cos nt = \lim_{n\to\infty} r^n \sin nt &= \lim_{n\to\infty} r^{n+1}\cos(n-1)t \\ &= \lim_{n\to\infty} r^{n+1}\sin(n-1)t = 0;\end{aligned}$$

consequently,

$$\sum_{k=0}^{\infty} r^k \cos kt = \lim_{n\to\infty} A_n = \frac{1 - r\cos t}{1 - 2r\cos t + r^2}$$

and, similarly

$$\sum_{k=0}^{\infty} r^k \sin kt = \lim_{n\to\infty} B_n = \frac{r\sin t}{1 - 2r\cos t + r^2}.$$

Example 3.2. *Prove the trigonometric identity*

$$\cos^n x = \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\cos((n-2k)x).$$

Solution. From the de Moivre's Formula we have

$$\begin{aligned}\cos^n x = \left(\frac{e^{ix}+e^{-ix}}{2}\right)^n &= \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}(e^{ix})^{n-k}(e^{-ix})^k \\ &= \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}e^{i(n-2k)x} \\ &= \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\cos((n-2k)x)\end{aligned}$$

To justify the last step note that we started with a real number, so the imaginary parts must cancel out and only the real part of each term contributes.

Example 3.3. *Let n be an odd positive integer and let z be a complex number such that*

$$z^{2^n-1}-1=0.$$

Evaluate

$$\prod_{k=0}^{n-1}\left(z^{2^k}+\frac{1}{z^{2^k}}-1\right).$$

Solution. Let

$$Z_n=\prod_{k=0}^{n-1}\left(z^{2^k}+\frac{1}{z^{2^k}}-1\right).$$

Repeatedly applying the identity

$$\left(a+\frac{1}{a}+1\right)\left(a+\frac{1}{a}-1\right)=\left(a+\frac{1}{a}\right)^2-1=a^2+\frac{1}{a^2}+1,$$

we have that

$$\begin{aligned}\left(z+\frac{1}{z}+1\right)Z_n&=\left(z^2+\frac{1}{z^2}+1\right)\left(z^2+\frac{1}{z^2}-1\right)\cdots\left(z^{2^{n-1}}+\frac{1}{z^{2^{n-1}}}+1\right)\\&=\left(z^{2^n}+\frac{1}{z^{2^n}}+1\right).\end{aligned}$$

However, from the given condition we have that $z^{2^n}=z$. Thus,

$$\left(z+\frac{1}{z}+1\right)Z_n=\left(z+\frac{1}{z}+1\right).$$

Now, since the roots of $z+\dfrac{1}{z}+1$ are primitive 6-th roots of unity, they are not (2^n-1)st roots of unity, and hence

$$z+\frac{1}{z}+1\neq 0.$$

Hence, it follows that $Z_n=1$.

Example 3.4. *Prove that for all $n \in \mathbb{N}$ the following is true:*

$$2^n \prod_{k=1}^{n} \sin \frac{k\pi}{2n+1} = \sqrt{2n+1}.$$

Solution. First we prove the following:

$$\prod_{k=1}^{n-1} \sin \frac{k\pi}{n} = n \cdot 2^{1-n}.$$

Let $\omega = e^{\frac{i\pi}{n}}$. Since $\sin x = \dfrac{e^{ix} - e^{-ix}}{2i}$ the product is then equal to

$$\left(\frac{\omega - \omega^{-1}}{2i}\right)\left(\frac{\omega^2 - \omega^{-2}}{2i}\right)\left(\frac{\omega^3 - \omega^{-3}}{2i}\right)\cdots\left(\frac{\omega^{n-1} - \omega^{-(n-1)}}{2i}\right)$$

$$= (1-\omega^{-2})(1-\omega^{-4})(1-\omega^{-6})\cdots(1-\omega^{-2(n-1)})\omega^{n(n-1)/2}(2i)^{-(n-1)}.$$

Since $\omega^{n/2} = i$, it remains to show that

$$(1-\omega^{-2})(1-\omega^{-4})(1-\omega^{-6})\cdots(1-\omega^{-2(n-1)}) = n.$$

Consider the polynomial given by

$$P(x) = (x-\omega^{-2})(x-\omega^{-4})(x-\omega^{-6})\cdots(x-\omega^{-2(n-1)}).$$

We know its roots are all complex n-th roots of unity except 1, so we must have

$$P(x) = \frac{x^n - 1}{x - 1} = 1 + x + x^2 + \cdots + x^{n-1}.$$

Putting $x = 1$ completes the proof of the product formula.
Now we may proceed with the problem: We may write the product formula we just proved as

$$\prod_{k=1}^{n} \sin \frac{k\pi}{2n+1} \cdot \prod_{k=n+1}^{2n} \sin \frac{k\pi}{2n+1} = \prod_{k=1}^{2n} \sin \frac{k\pi}{2n+1} = (2n+1)2^{-2n}.$$

Using $\sin(\pi - x) = \sin x$ on each term in the second factor on the left, gives

$$\left(\prod_{k=1}^{n} \sin\frac{k\pi}{2n+1}\right)^2 = (2n+1)2^{-2n}.$$

Solving for the product gives

$$\prod_{k=1}^{n} \sin\frac{k\pi}{2n+1} = 2^{-n}\sqrt{2n+1}$$

as required.
Prove in a similar manner that

$$\prod_{k=1}^{n-1} \sin\frac{k\pi}{2n} = \prod_{k=1}^{n-1} \cos\frac{k\pi}{2n} = \frac{\sqrt{n}}{2^{n-1}}.$$

Example 3.5. *For each positive integer n prove that*

$$\cot^2\frac{\pi}{2n+1} + \cot^2\frac{2\pi}{2n+1} + \cdots + \cot^2\frac{n\pi}{2n+1} = \frac{n(2n-1)}{3}.$$

Solution. Using de Moivre's Formula we have

$$(\cos a + i\sin a)^m = \cos ma + i\sin ma.$$

Expanding the right hand side we get

$$\cos ma = \cos^m a - \binom{m}{2}\cos^{m-2} a\sin^2 a + \binom{m}{4}\cos^{m-4} a\sin^4 a - \cdots$$

$$\sin ma = \binom{m}{1}\cos^{m-1} a\sin a - \binom{m}{3}\cos^{m-3} a\sin^3 a - \cdots$$

Let us replace m by $2n+1$:

$$\begin{aligned}\cos(2n+1)a &= \cos^{2n+1} a - \binom{2n+1}{2}\cos^{2n-1} a\sin^2 a \\ &\quad + \cdots + \binom{2n+1}{2n}\cos a\sin^{2n} a \\ \sin(2n+1)a &= \binom{2n+1}{1}\cos^{2n} a\sin a - \binom{2n+1}{3}\cos^{2n-2} a\sin^3 a \\ &\quad - \cdots + \sin^{2n+1} a.\end{aligned}$$

Hence

$$\frac{\sin(2n+1)a}{\sin^{2n+1} a} = \binom{2n+1}{1}(\cot^2 a)^n - \binom{2n+1}{3}(\cot^2 a)^{n-1} - \cdots + 1.$$

The equation

$$\frac{\sin(2n+1)a}{\sin^{2n+1} a} = 0$$

viewed as a polynomial equation in $x = \cot^2 a$ has solutions

$$x_k = \cot^2 \frac{k\pi}{2n+1} = \cot^2 \frac{(2n+1-k)\pi}{2n+1},$$

because $\alpha_k = \dfrac{k\pi}{2n+1}$ are the solutions to the equation $\sin(2n+1)a = 0$. Using the relation between the solutions and the coefficients, we obtain

$$x_1 + x_2 + \cdots + x_n = \frac{\binom{2n+1}{3}}{\binom{2n+1}{1}}.$$

Thus

$$\cot^2 \frac{\pi}{2n+1} + \cot^2 \frac{2\pi}{2n+1} + \cdots + \cot^2 \frac{n\pi}{2n+1} = \frac{n(2n-1)}{3},$$

and we are done.

Example 3.6. *Prove that*

$$\frac{1}{\sin^2 \dfrac{\pi}{4n+2}} + \frac{1}{\sin^2 \dfrac{3\pi}{4n+2}} + \cdots + \frac{1}{\sin^2 \dfrac{(2n-1)\pi}{4n+2}} = 2n(n+1)$$

for every positive integer n.

Solution. Let

$$P(z) = (z+i)^{2n+1} = \sum_{k=0}^{2n+1} \binom{2n+1}{k} i^{2n+1-k} z^k.$$

For real z, the real part of $P(z)$ is:

$$\operatorname{Re}(P(z)) = \sum_{k=0}^{n} \binom{2n+1}{2k+1} (-1)^{n-k} z^{2k+1} = zQ(z^2)$$

where

$$Q(z) = \sum_{k=0}^{n} (-1)^{n-k} \binom{2n+1}{2k+1} z^k.$$

Now, for $p \in \{0, 1, \ldots, n-1\}$:

$$\begin{aligned}
P\left(\cot \frac{2p+1}{4n+2}\pi\right) &= \left(\cot \frac{2p+1}{4n+2}\pi + i\right)^{2n+1} \\
&= \frac{\left(\cos \frac{2p+1}{4n+2}\pi + i \sin \frac{2p+1}{4n+2}\pi\right)^{2n+1}}{\sin^{2n+1} \frac{2p+1}{4n+2}\pi} \\
&= \frac{\cos \frac{2p+1}{2}\pi + i \sin \frac{2p+1}{2}\pi}{\sin^{2n+1} \frac{2p+1}{4n+2}\pi} \\
&= \frac{(-1)^p}{\sin^{2n+1} \frac{2p+1}{4n+2}\pi} i
\end{aligned}$$

hence we have

$$\operatorname{Re}\left(P\left(\cot \frac{2p+1}{4k+2}\pi\right)\right) = 0.$$

Consequently,

$$\cot \frac{2p+1}{4k+2}\pi Q\left(\cot^2 \frac{2p+1}{4n+2}\pi\right) = 0$$

for all $p \in \{0, 1, \ldots, n-1\}$ and then

$$Q\left(\cot^2 \frac{2p+1}{4n+2}\pi\right) = 0, \ \forall\, p \in \{0, 1, \ldots, n-1\}.$$

This shows that the numbers $\cot^2 \dfrac{2p+1}{4n+2}\pi$ with $p \in \{0, 1, \dots, n-1\}$ are the n distinct real roots of

$$Q(z) = \sum_{k=0}^{n} \binom{2n+1}{2k+1} (-1)^{n-k} z^k$$

so that their sum is

$$\sum_{p=0}^{n-1} \cot^2 \frac{2p+1}{4n+2}\pi = -\frac{-\binom{2n+1}{2n-1}}{\binom{2n+1}{2n+1}} = n(2n+1)$$

and since

$$\frac{1}{\sin^2 x} = \cot^2 x + 1,$$

we get

$$\sum_{p=0}^{n-1} \frac{1}{\sin^2 \dfrac{2p+1}{4n+2}\pi} = n(2n+1) + n = 2n(n+1),$$

as desired.

Example 3.7. *Prove the identity*

$$\sum_{l \equiv 0 \,(\mathrm{mod}\, k)} \binom{n}{l} = \binom{n}{0} + \binom{n}{k} + \binom{n}{2k} + \cdots = \frac{2^n}{k} \sum_{j=0}^{k-1} \cos^n \frac{j\pi}{k} \cos \frac{nj\pi}{k}.$$

Solution. Let $\varepsilon_0, \varepsilon_1, \varepsilon_2, \dots, \varepsilon_{k-1}$ be the kth roots of unity, that is

$$\varepsilon_j = \cos \frac{2j\pi}{k} + i \sin \frac{2j\pi}{k}, \ j = 0, 1, \dots, k-1.$$

An important fact about roots of unity is that the sum $\varepsilon_1^s + \varepsilon_2^s + \cdots + \varepsilon_k^s$ is equal to k if k divides s, and 0 otherwise. Indeed, if k divides s, then each term in this sum is 1 and the sum is k. If k does not divide s, then the sum is a geometric series and we have

$$\varepsilon_1^s + \varepsilon_2^s + \cdots + \varepsilon_k^s = \varepsilon_1^s + \varepsilon_1^{2s} + \cdots + \varepsilon_1^{ks} = \frac{\varepsilon_1^s(1 - \varepsilon_1^{ks})}{1 - \varepsilon_1^s} = 0.$$

We have

$$\sum_{j=0}^{k-1}(1+\varepsilon_j)^n = \sum_{s=0}^{n}\binom{n}{s}\left(\sum_{j=0}^{k-1}\varepsilon_j^s\right) = k\sum_{j=0}^{\lfloor \frac{n}{k}\rfloor}\binom{n}{jk}.$$

Since

$$1+\varepsilon_j = 2\cos\frac{j\pi}{k}\left(\cos\frac{j\pi}{k}+i\sin\frac{j\pi}{k}\right),$$

it follows from the de Moivre formula that

$$\sum_{j=0}^{k-1}(1+\varepsilon_j)^n = 2^n\sum_{j=0}^{k-1}\cos^n\frac{j\pi}{k}\left(\cos\frac{nj\pi}{k}+i\sin\frac{nj\pi}{k}\right).$$

Matching the real parts we get the desired result.

Example 3.8. *Let n be a positive integer, $\varepsilon_0,\ldots,\varepsilon_{n-1}$ be the nth roots of unity, and a, b be complex numbers. Evaluate the product*

$$\prod_{k=0}^{n-1}(a+b\varepsilon_k^2).$$

Solution. By definition, $\varepsilon_0,\varepsilon_1,\ldots,\varepsilon_{n-1}$ are roots of the equation $x^n-1=0$, over the complex numbers. Therefore by the factor theorem we have

$$\prod_{k=0}^{n-1}(x-\varepsilon_k) = x^n-1. \tag{1}$$

Let z be a solution of the equation $z^2 = -\frac{a}{b}$.
Then, in an effort to use (1), we write

$$\prod_{k=0}^{n-1}(a+b\varepsilon_k^2) = \prod_{k=0}^{n-1}(-b)\left(\frac{a}{-b}-\varepsilon_k^2\right) = (-b)^n\prod_{k=0}^{n-1}(z^2-\varepsilon_k^2)$$

$$= (-b)^n\prod_{k=0}^{n-1}(z-\varepsilon_k)\prod_{k=0}^{n-1}(z+\varepsilon_k)$$

$$= b^n \prod_{k=0}^{n-1}(z-\varepsilon_k)\prod_{k=0}^{n-1}((-z)-\varepsilon_k)$$

$$= b^n(z^n-1)((-z)^n-1) = b^n[(-1)^n z^{2n} - z^n((-1)^n+1)+1].$$

To simplify the $(-1)^n$ terms in the above expression, we consider cases of the parity of n.

If n is even, that is $n=2m$ for some positive integer m, then we have

$$\prod_{k=0}^{n-1}(a+b\varepsilon_k^2) = b^{2m}(z^{4m}-2z^{2m}+1) = b^{2m}\left(\left(-\frac{a}{b}\right)^{2m} - 2\left(-\frac{a}{b}\right)^{m}+1\right)$$

$$= a^{2m} - 2a^m(-b)^m + b^{2m} = (a^m-(-b)^m)^2.$$

If n is odd then

$$\prod_{k=0}^{n-1}(a+b\varepsilon_k^2) = b^n(-z^{2n}+1) = b^n\left(-\left(-\frac{a}{b}\right)^n+1\right) = a^n+b^n.$$

Thus,

$$\prod_{k=0}^{n-1}(a+b\varepsilon_k^2) = \begin{cases} \left(a^{\frac{n}{2}}-(-b)^{\frac{n}{2}}\right)^2 & \text{if } n \text{ is even} \\ a^n+b^n & \text{if } n \text{ is odd.}\end{cases}$$

Example 3.9. *Prove that*

$$\sum_{k=0}^{29}\tan\left((6k+1)\frac{\pi}{180}\right) = -30\sqrt{3}.$$

Solution. For easier writing, let

$$a_p = (12p+1)\frac{\pi}{180} \quad \text{and} \quad b_p = (12p+7)\frac{\pi}{180}.$$

Note that $\tan a_p$ and $\tan b_p$ when $p \in \{0,1,2,3,4,5,6,7,8,9,10,11,12,13,14\}$ are pairwise distinct.

Let

$$P(z) = (1+iz)^{30} = \left(\sum_{k=0}^{15}\binom{30}{2k}(-1)^k z^{2k}\right) + i\left(\sum_{k=0}^{14}\binom{30}{2k+1}(-1)^k z^{2k+1}\right)$$

be a polynomial of degree 30.
We have

$$\frac{\mathrm{Re}(P(z))}{\mathrm{Im}(P(z))} = \frac{\displaystyle\sum_{k=0}^{15} \binom{30}{2k} (-1)^k z^{2k}}{\displaystyle\sum_{k=0}^{14} \binom{30}{2k+1} (-1)^k z^{2k+1}}$$

where $\mathrm{Re}(w)$ and $\mathrm{Im}(w)$ denote the real part and the imaginary part of the complex number w.
We have

$$P(\tan a_p) = \frac{(\cos a_p + i \sin a_p)^{30}}{(\cos a_p)^{30}} = \frac{\cos 30a_p + i \sin 30a_p}{(\cos a_p)^{30}} = \frac{\sqrt{3} + i}{2(\cos a_p)^{30}}$$

meaning that

$$\frac{\mathrm{Re}(P(\tan a_p))}{\mathrm{Im}(P(\tan a_p))} = \sqrt{3},$$

that is

$$\frac{\displaystyle\sum_{k=0}^{15} \binom{30}{2k} (-1)^k (\tan a_p)^{2k}}{\displaystyle\sum_{k=0}^{14} \binom{30}{2k+1} (-1)^k (\tan a_p)^{2k+1}} = \sqrt{3}.$$

Thus

$$\sum_{k=0}^{15} \binom{30}{2k} (-1)^k (\tan a_p)^{2k} - \sqrt{3} \sum_{k=0}^{14} \binom{30}{2k+1} (-1)^k (\tan a_p)^{2k+1} = 0.$$

Now, for b_p, we can compute similarly

$$P(\tan b_p) = \frac{(\cos b_p + i \sin b_p)^{30}}{(\cos b_p)^{30}} = \frac{\cos 30b_p + i \sin 30b_p}{(\cos b_p)^{30}} = -\frac{\sqrt{3} + i}{2(\cos b_p)^{30}},$$

therefore

$$\frac{\mathrm{Re}(P(\tan b_p))}{\mathrm{Im}(P(\tan b_p))} = \sqrt{3},$$

and, again

$$\sum_{k=0}^{15}\binom{30}{2k}(-1)^k(\tan b_p)^{2k}-\sqrt{3}\sum_{k=0}^{14}\binom{30}{2k+1}(-1)^k(\tan b_p)^{2k+1}=0.$$

Thus we have seen that the polynomial

$$\begin{aligned}Q(x)&=\sum_{k=0}^{15}\binom{30}{2k}(-1)^k x^{2k}-\sqrt{3}\sum_{k=0}^{14}\binom{30}{2k+1}(-1)^k x^{2k+1}\\&=-x^{30}-30\sqrt{3}x^{29}+\cdots\end{aligned}$$

has degree 30 and the numbers $\tan a_p$ and $\tan b_p$, $p=0,1,\dots 14$ are 30 distinct roots of $Q(x)$. So

$$\sum_{p=0}^{14}\tan(12p+1)\frac{\pi}{180}+\sum_{p=0}^{14}\tan(12p+7)\frac{\pi}{180}$$

is the sum of roots of Q and, by Viete's relations, it is $-30\sqrt{3}$. Consequently

$$\sum_{k=0}^{29}\tan\left((6k+1)\frac{\pi}{180}\right)=-30\sqrt{3}$$

follows, because the numbers a_p and b_p together are precisely the numbers $(6k+1)\pi/180$, $k=0,1,\dots,29$.

Example 3.10. *Find the sum*

$$\sum_{j=1}^{89}\tan^2\frac{j\pi}{180}.$$

Solution. We have

$$(\cos x+i\sin x)^n=\cos nx+i\sin nx$$

$$\Rightarrow (1+i\tan x)^n=\frac{1}{\cos^n x}(\cos nx+i\sin nx),$$

or

$$\sum_{k=0}^{n} \binom{n}{k} i^k \tan^k x = \frac{1}{\cos^n x}(\cos nx + i \sin nx)$$

for any positive integer n. Replacing here $n = 180$ and $x = \dfrac{j\pi}{180}$ yields

$$\sum_{k=0}^{n} \binom{n}{k} i^k \tan^k \frac{j\pi}{180} = \frac{1}{\cos^n \frac{j\pi}{180}}(\cos j\pi + i \sin j\pi) = \frac{(-1)^j}{\cos^n \frac{j\pi}{180}}$$

for $j = 1, 2, \ldots, 89$.
Thus the imaginary part of the sum from the left is 0, that is

$$\sum_{p=0}^{89} (-1)^p \binom{180}{2p+1} \tan^{2p+1} \frac{j\pi}{180} = 0$$
$$\Leftrightarrow \sum_{p=0}^{89} (-1)^p \binom{180}{2p+1} \tan^{2p} \frac{j\pi}{180} = 0,$$

since $\tan \dfrac{j\pi}{180} \neq 0$ for $1 \leq j \leq 89$. Thus we found that the distinct numbers $\tan^2 \dfrac{j\pi}{180}$, $j = 1, 2, \ldots, 89$ are the roots of the 89th degree polynomial

$$\sum_{p=0}^{89} (-1)^p \binom{180}{2p+1} x^p,$$

therefore their sum is

$$\sum_{j=1}^{89} \tan^2 \frac{j\pi}{180} = -\frac{\binom{180}{177}}{-\binom{180}{179}} = \frac{15931}{3}.$$

Chapter 4

The Abel Summation Formula

The next result we want to present is the famous Abel Summation Formula. Let $a_1, a_2, \ldots, a_n$, and $b_1, b_2, \ldots, b_n$, be two finite sequences of numbers. Then

$$\begin{aligned} a_1b_1 + a_2b_2 + \cdots + a_nb_n = (a_1 - a_2)b_1 + (a_2 - a_3)(b_1 + b_2) + \cdots \\ + (a_{n-1} - a_n)(b_1 + \cdots + b_{n-1}) + a_n(b_1 + \cdots + b_n). \end{aligned}$$

Example 4.1. *For*

$$f(t) = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{t}$$

(with positive integer t) prove that

$$\sum_{k=1}^{n}(2k+1)f(k) = (n+1)^2 f(n) - \frac{n(n+1)}{2}.$$

Solution. Define $f(0) = 0$ and $g(k) = 2k + 1$ for $0 \leq k \leq n$. By the Abel summation formula we have

$$\sum_{k=1}^{n}(2k+1)f(k) = \sum_{k=0}^{n} g(k)f(k) = f(n)\sum_{k=0}^{n} g(k) - \sum_{j=0}^{n-1}(f(j+1) - f(j))\sum_{k=0}^{j} g(k).$$

But

$$\sum_{k=0}^{j} g(k) = \sum_{k=0}^{j} (2k+1) = (j+1)^2, \quad \text{and} \quad f(j+1) - f(j) = \frac{1}{j+1},$$

so

$$\sum_{k=0}^{n} g(k) = (n+1)^2$$

and

$$\sum_{j=0}^{n-1} (f(j+1) - f(j)) \sum_{k=0}^{j} g(k) = \sum_{j=0}^{n-1} (j+1) = \frac{n(n+1)}{2},$$

and we are done.

Example 4.2. (Abel's inequality) *Let $b_1 \geq \cdots \geq b_n$ be a nonincreasing sequence of nonnegative real numbers and let $a_1, \ldots, a_n$ be real numbers. Assume that m, M are real numbers such that*

$$m \leq a_1 + \cdots + a_k \leq M$$

for all $1 \leq k \leq n$. Prove that

$$b_1 m \leq a_1 b_1 + \cdots + a_n b_n \leq b_1 M.$$

Solution. Let $A_k = a_1 + \cdots + a_k$. By Abel's summation formula we obtain (with the convention $b_{n+1} = 0$)

$$a_1 b_1 + \cdots + a_n b_n = \sum_{k=1}^{n} A_k (b_k - b_{k+1}).$$

Since by assumption $b_k - b_{k+1} \geq 0$ and $m \leq A_k \leq M$, we obtain

$$\sum_{k=1}^{n} m(b_k - b_{k+1}) \leq \sum_{k=1}^{n} A_k (b_k - b_{k+1}) \leq \sum_{k=1}^{n} M(b_k - b_{k+1}).$$

Both sums appearing in the extreme terms are telescopic and reduce to mb_1, respectively Mb_1 (we recall that $b_{n+1} = 0$). The result follows.

Example 4.3. *Let $\phi : \mathbb{N}^* \to \mathbb{N}^*$ be an injective function. Prove that for all $n \in \mathbb{N}^*$*

$$\sum_{k=1}^{n} \frac{\phi(k)}{k^2} \geq \sum_{k=1}^{n} \frac{1}{k}.$$

Solution. The inequality becomes

$$\sum_{k=1}^{n} \frac{1}{k}\left(\frac{\phi(k)}{k} - 1\right) \geq 0.$$

If we set $\lambda_k = \dfrac{\phi(k)}{k}$, the the injectivity of ϕ implies $\lambda_1\lambda_2 \cdots \lambda_k \geq 1$ for all k. From the AM-GM inequality we obtain

$$\sum_{i=1}^{k} \lambda_k \geq k\sqrt[k]{\lambda_1\lambda_2\cdots\lambda_k} \geq k.$$

Applying the Abel summation formula yields

$$\sum_{k=1}^{n} \frac{1}{k}(\lambda_k - 1) = \sum_{k=1}^{n}\left(\frac{1}{k-1} - \frac{1}{k}\right)(\lambda_1 + \lambda_2 + \cdots + \lambda_k - k)$$
$$+ \frac{1}{n}(\lambda_1 + \lambda_2 + \cdots + \lambda_n - n).$$

Each term of the sum is positive so we are done.

Example 4.4. *Let $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n$ be positive real numbers such that:*
(i) $x_1y_1 < x_2y_2 < \cdots < x_ny_n$,
(ii) $x_1 + x_2 + \cdots + x_k \geq y_1 + y_2 + \cdots + y_k,\ 1 \leq k \leq n.$
Prove that

$$\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n} \leq \frac{1}{y_1} + \frac{1}{y_2} + \cdots + \frac{1}{y_n}.$$

Solution. Let

$$S_k = (x_1 - y_1) + (x_2 - y_2) + \cdots + (x_k - y_k) \quad \text{and} \quad z_k = \frac{1}{x_ky_k}.$$

Then, we have $S_k \geq 0$ and $z_k - z_{k+1} > 0$, for any $k = 1, 2, \ldots, n-1$. It follows

$$\begin{aligned}
&\left(\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}\right) - \left(\frac{1}{y_1} + \frac{1}{y_2} + \cdots + \frac{1}{y_n}\right) \\
&= \left(\frac{1}{x_1} - \frac{1}{y_1}\right) + \left(\frac{1}{x_2} - \frac{1}{y_2}\right) + \cdots + \left(\frac{1}{x_n} - \frac{1}{y_n}\right) \\
&= \frac{y_1 - x_1}{x_1 y_1} + \frac{y_2 - x_2}{x_2 y_2} + \cdots + \frac{y_n - x_n}{x_n y_n} \\
&= -S_1 z_1 - (S_2 - S_1) z_2 - \cdots - (S_n - S_{n-1}) z_n \\
&= -S_1(z_1 - z_2) - S_2(z_2 - z_3) - \cdots - S_{n-1}(z_{n-1} - z_n) - S_n z_n \leq 0,
\end{aligned}$$

with equality if and only if $S_k = 0$, $k = 1, 2, \ldots, n$, that is, $x_k = y_k$, $k = 1, 2, \ldots, n$.

Example 4.5. *Let $x_1 \geq x_2 \geq \cdots \geq x_n > 0$ and $y_1, y_2, \ldots, y_n > 0$ be two sequences of positive numbers such that*

$$y_1 y_2 \cdots y_k \geq x_1 x_2 \cdots x_k \text{ for all } 1 \leq k \leq n.$$

Prove that

$$y_1 + y_2 + \cdots + y_n \geq x_1 + x_2 + \cdots + x_n.$$

Solution. Combining the hypothesis and the AM-GM inequality, we obtain

$$\frac{y_1}{x_1} + \frac{y_2}{x_2} + \cdots + \frac{y_k}{x_k} \geq k \sqrt[k]{\frac{y_1 y_2 \cdots y_k}{x_1 x_2 \cdots x_k}} \geq k.$$

On the other hand, Abel's summation formula yields (we let $x_{n+1} = 0$)

$$y_1 + y_2 + \cdots + y_n = \sum_{k=1}^{n} \frac{y_k}{x_k} \cdot x_k = \sum_{k=1}^{n} (x_k - x_{k+1}) \left(\frac{y_1}{x_1} + \frac{y_2}{x_2} + \cdots + \frac{y_k}{x_k}\right).$$

By assumption $x_k - x_{k+1} \geq 0$ for $1 \leq k \leq n$, which combined with the previous observation yields

$$\begin{aligned}
y_1 + y_2 + \cdots + y_n &= \sum_{k=1}^{n} (x_k - x_{k+1}) \left(\frac{y_1}{x_1} + \frac{y_2}{x_2} + \cdots + \frac{y_k}{x_k}\right) \\
&\geq \sum_{k=1}^{n} k(x_k - x_{k+1}) = x_1 + x_2 + \cdots + x_n
\end{aligned}$$

and finishes the proof (for the last equality Abel's summation formula was used once more).

Example 4.6. *Let $a_1, a_2, a_3, \ldots, a_n$ be positive real numbers such that*

$$a_1 a_2 \cdots a_k \geq \frac{1}{(2k)!}$$

for all $1 \leq k \leq n$. Prove that

$$a_1 + a_2 + \cdots + a_n \geq \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}.$$

Solution. We choose $y_k = a_k$ and

$$x_k = \frac{1}{(2k-1)2k} = \frac{1}{2k-1} - \frac{1}{2k}$$

in the previous exercise. We clearly have $x_1 \geq x_2 \geq \cdots \geq x_n$ and, by hypothesis,

$$y_1 y_2 \cdots y_k = a_1 a_2 \cdots a_k \geq \frac{1}{(2k)!} = \frac{1}{1 \cdot 2} \cdot \frac{1}{3 \cdot 4} \cdots \frac{1}{(2k-1)2k} = x_1 x_2 \cdots x_k$$

for every $1 \leq k \leq n$.
So the result from the previous example applies, and we deduce

$$y_1 + y_2 + \cdots + y_n \geq x_1 + x_2 + \cdots + x_n,$$

that is,

$$a_1 + a_2 + \cdots + a_n \geq \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{2n-1} - \frac{1}{2n}$$

$$= \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{2n-1} + \frac{1}{2n} - 2\left(\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2n}\right)$$

$$= \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{2n-1} + \frac{1}{2n} - \left(\frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n}\right)$$

$$= \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+n},$$

as required.

Example 4.7. *Let $a_1, \ldots, a_n$ be real numbers such that $a_1 + \cdots + a_k \le k$ for $1 \le k \le n$. Prove that*

$$\frac{a_1}{1} + \frac{a_2}{2} + \cdots + \frac{a_n}{n} \le \frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n}.$$

Solution. Let $A_k = a_1 + \cdots + a_k$, thus by assumption $A_k \le k$ for $1 \le k \le n$. Abel's summation formula yields

$$\sum_{k=1}^{n} \frac{a_k}{k} = \frac{A_n}{n} + \sum_{j=1}^{n-1} A_j \left(\frac{1}{j} - \frac{1}{j+1}\right) \le 1 + \sum_{j=1}^{n-1} j \left(\frac{1}{j} - \frac{1}{j+1}\right).$$

Applying once more Abel's summation formula for the last sum, we obtain

$$1 + \sum_{j=1}^{n-1} j \left(\frac{1}{j} - \frac{1}{j+1}\right) = \sum_{k=1}^{n} \frac{1}{k}.$$

Thus

$$\frac{a_1}{1} + \frac{a_2}{2} + \cdots + \frac{a_n}{n} \le \frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n},$$

as desired.

Example 4.8. *For all $a_1, a_2, \ldots, a_n$, $b_1, b_2, \ldots, b_n$ such that $0 < a_1 \le a_2 \le \cdots \le a_n$, $0 < b_1 \le b_2 \le \cdots \le b_n$ and*

$$\sum_{i=1}^{k} a_i^2 \le \sum_{i=1}^{k} b_i^2, \ \ k = 1, 2, \ldots, n$$

we have

$$\sum_{i=1}^{n} a_i \le \sum_{i=1}^{n} b_i.$$

Solution. By using Abel's summation formula we have

$$\sum_{i=1}^{n} (b_i - a_i) = \sum_{i=1}^{n} \frac{1}{b_i + a_i} \cdot (b_i^2 - a_i^2)$$

$$= \frac{1}{b_n + a_n} \cdot \sum_{i=1}^{n} (b_i^2 - a_i^2) + \sum_{k=1}^{n-1} \left(\frac{1}{b_k + a_k} - \frac{1}{b_{k+1} + a_{k+1}}\right) \sum_{i=1}^{k} (b_i^2 - a_i^2) \ge 0.$$

Example 4.9. *Let $b_1 \geq b_2 \geq \cdots \geq b_n \geq 0$ be a nonincreasing sequence of nonnegative real numbers. Let $a_1, a_2 \ldots, a_n$ be real numbers such that*

$$a_1 + a_2 + \cdots + a_k \geq b_1 + b_2 + \cdots + b_k \text{ for all } 1 \leq k \leq n.$$

Prove that

$$a_1^2 + a_2^2 + \cdots + a_n^2 \geq b_1^2 + b_2^2 + \cdots + b_n^2.$$

Solution. We will prove the stronger inequality

$$a_1 b_1 + a_2 b_2 + \cdots + a_n b_n \geq b_1^2 + b_2^2 + \cdots + b_n^2.$$

To see that this implies the desired result, one can use either the Cauchy-Schwarz inequality or add up the inequalities

$$a_k^2 + b_k^2 \geq 2a_k b_k \text{ for } 1 \leq k \leq n.$$

For $S_k = a_1 + a_2 + \cdots + a_k$ Abel's summation formula yields

$$a_1 b_1 + \cdots + a_n b_n = (b_1 - b_2)S_1 + (b_2 - b_3)S_2 + \cdots + (b_{n-1} - b_n)S_{n-1} + b_n S_n.$$

By assumption $b_1 - b_2, \ldots, b_{n-1} - b_n, b_n$ are nonnegative and

$$S_k \geq b_1 + b_2 + \cdots + b_k \text{ for } 1 \leq k \leq n,$$

thus

$$\begin{gathered} a_1 b_1 + \cdots + a_n b_n \geq (b_1 - b_2)b_1 + (b_2 - b_3)(b_1 + b_2) \\ + \cdots + (b_{n-1} - b_n)(b_1 + \cdots + b_{n-1}) + b_n(b_1 + \cdots + b_n). \end{gathered}$$

Applying once more Abel's summation formula to the right-hand side, we obtain

$$\begin{gathered} (b_1 - b_2)b_1 + (b_2 - b_3)(b_1 + b_2) + \cdots + (b_{n-1} - b_n)(b_1 + \cdots + b_{n-1}) \\ + b_n(b_1 + \cdots + b_n) = b_1^2 + b_2^2 + \cdots + b_n^2, \end{gathered}$$

yielding therefore the desired inequality

$$a_1 b_1 + a_2 b_2 + \cdots + a_n b_n \geq b_1^2 + b_2^2 + \cdots + b_n^2.$$

Example 4.10. *Let $a_1, a_2, \ldots, a_n$ be positive real numbers such that*

$$\sum_{i=1}^{k} a_i \geq \sqrt{k} \text{ for every } k = 1, 2, \ldots, n.$$

Prove that

$$\sum_{k=1}^{n} a_k^2 > \frac{1}{4}\left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right).$$

Solution. We use the previous example, by choosing $b_k = \sqrt{k} - \sqrt{k-1}$ for $1 \leq k \leq n$. We have, indeed,

$$b_k = \frac{1}{\sqrt{k} + \sqrt{k-1}} > \frac{1}{\sqrt{k+1} + \sqrt{k}} = b_{k+1}$$

for $1 \leq k \leq n-1$ and, by hypothesis,

$$\sum_{i=1}^{k} a_i \geq \sqrt{k} = \sum_{i=1}^{k} b_i$$

for $1 \leq k \leq n$. Thus the conditions are fulfilled to apply the result from the previous example, and to infer that

$$\sum_{k=1}^{n} a_k^2 \geq \sum_{k=1}^{n} b_k^2.$$

But we also have the inequalities

$$b_k = \sqrt{k} - \sqrt{k-1} = \frac{1}{\sqrt{k} + \sqrt{k-1}} > \frac{1}{2\sqrt{k}}$$

for $k = 1, 2, \ldots, n$. Thus,

$$\sum_{k=1}^{n} a_k^2 \geq \sum_{k=1}^{n} b_k^2 > \sum_{k=1}^{n} \left(\frac{1}{2\sqrt{k}}\right)^2 = \frac{1}{4} \sum_{k=1}^{n} \frac{1}{k},$$

which is precisely what we intended to prove.

Example 4.11. (IMO Longlist 1976) *Let $a_0, a_1, \ldots, a_n, a_{n+1}$ be a sequence of real numbers satisfying the following conditions:*

$$a_0 = a_{n+1} = 0, \; |a_{k-1} - 2a_k + a_{k+1}| \leq 1 \; (k = 1, 2, \ldots, n).$$

Prove that

$$|a_k| \leq \frac{k(n+1-k)}{2} \; (k = 0, 1, \ldots, n+1).$$

Solution. If we substitute $a_{k-1} - a_k = b_{k-1}$ so that the hypothesis becomes $|b_k - b_{k+1}| \leq 1$, we have the following two identities (actually some Abel type formulas):

$$-a_k = \sum_{i=0}^{k-1} (i+1)(b_i - b_{i+1}) + kb_k$$

and

$$a_k = \sum_{i=k}^{n-1} (b_{i+1} - b_i)(n-i) + (n-k+1)b_k$$

We have

$$\begin{aligned}
&(n+1)|a_k| = |ka_k + (n-k+1)a_k| \\
&= \left| k \left[\sum_{i=k}^{n-1} (b_{i+1} - b_i)(n-i) \right] - (n-k+1) \left[\sum_{i=0}^{k-1} (i+1)(b_i - b_{i+1}) \right] \right| \\
&\leq k[1 + 2 + \cdots + n - k] + (n-k+1)[1 + 2 + \cdots + k] \\
&= \frac{k(n-k+1)(n+1)}{2}.
\end{aligned}$$

Example 4.12. *Let $a_1 \geq a_2 \geq \cdots \geq a_n$ be a nonincreasing sequence of real numbers. Let $x_1, \ldots, x_n$ be real numbers such that*

$$x_1 + x_2 + \cdots + x_n = 0 \quad \textit{and} \quad |x_1| + |x_2| + \cdots + |x_n| = 1.$$

Prove that

$$|a_1 x_1 + \cdots + a_n x_n| \leq \frac{a_1 - a_n}{2}.$$

Solution. Define $S_k = x_1 + \cdots + x_k$, then Abel's summation formula combined with the triangle inequality and the equality $S_n = 0$ yield

$$|a_1x_1 + \cdots + a_nx_n| = \left|\sum_{k=1}^{n-1}(a_k - a_{k+1})S_k\right| \leq \sum_{k=1}^{n-1}|S_k| \cdot (a_k - a_{k+1}).$$

Since

$$\sum_{k=1}^{n-1}(a_k - a_{k+1}) = a_1 - a_n,$$

it suffices to prove that $|S_k| \leq \frac{1}{2}$ for $1 \leq k \leq n-1$. On the other hand

$$|S_k| \leq \max\left(\sum_{j\leq k, x_j \geq 0} x_j, \sum_{j \leq k, x_j < 0}(-x_j)\right),$$

since $|a - b| \leq \max(a, b)$ for $a, b \geq 0$. It suffices therefore to prove that

$$\sum_{j\leq k, x_j\geq 0} x_j \leq \frac{1}{2} \quad \text{and} \quad \sum_{j\leq k, x_j<0} (-x_j)) \leq \frac{1}{2}$$

for $1 \leq k \leq n$. For this we may assume that $k = n$ (as in this case the sums we are dealing with are maximal). But the hypotheses of the problem become

$$\sum_{x_j\geq 0} x_j - \sum_{x_j<0}(-x_j) = 0, \quad \sum_{x_j\geq 0} x_j + \sum_{x_j<0}(-x_j) = 1,$$

which yields

$$\sum_{x_j\geq 0} x_j = \sum_{x_j<0}(-x_j) = \frac{1}{2},$$

as needed.

Example 4.13. (Russia 2000) *Let* $-1 < x_1 < x_2 \cdots < x_n < 1$ *be real numbers such that*

$$x_1^{13} + x_2^{13} + \cdots + x_n^{13} = x_1 + x_2 + \cdots + x_n.$$

Prove that if $y_1 < y_2 < \cdots < y_n$ *are real numbers, then*

$$x_1^{13}y_1 + \cdots + x_n^{13}y_n < x_1y_1 + x_2y_2 + \cdots + x_ny_n.$$

Solution. By Abel's formula we have

$$\sum_{i=1}^{n} y_i(x_i^{13}-x_i) = (y_1-y_2)(x_1^{13}-x_1)+(y_2-y_3)(x_1^{13}+x_2^{13}-x_1-x_2)+\cdots$$
$$+(y_{n-1}-y_n)\left(\sum_{i=1}^{n-1} x_i^{13}-\sum_{i=1}^{n-1} x_i\right)+y_n\left(\sum_{i=1}^{n} x_i^{13}-\sum_{i=1}^{n} x_i\right).$$

Since $y_k - y_{k+1} < 0$ if we prove that

$$\sum_{i=1}^{k}(x_i^{13}-x_i) \geq 0 \quad \text{or} \quad \sum_{i=1}^{k} x_i(x_i^{12}-1) \geq 0$$

for all $k \leq n$ proof will finish.
If $x_k \leq 0$, then once again by Abel's formula we have

$$\sum_{i=1}^{k} x_i(x_i^{12}-1) = (x_1-x_2)c_1+(x_2-x_3)c_2+\cdots+(x_{k-1}-x_k)c_{k-1}+x_kc_k$$

where

$$c_r = \sum_{i=1}^{r}(x_i^{12}-1).$$

Since $-1 \leq x_i \leq 1$ we have

$$\sum_{i=1}^{r} x_i^{12} \leq r \quad \text{or} \quad c_r \leq 0.$$

Thus since we assumed $x_k \leq 0$, every term in above sum is non-negative.
If $x_k > 0$, then $x_i > 0$ for all $i \geq k+1$ and hence

$$\sum_{i=k+1}^{n} x_i \geq \sum_{i=k+1}^{n} x_i^{13}.$$

But since we assumed

$$x_1^{13}+x_2^{13}+\cdots+x_n^{13} = x_1+x_2+\cdots+x_n$$

this implies

$$\sum_{i=1}^{k} x_i^{13} \geq \sum_{i=1}^{k} x_i.$$

Example 4.14. *Let A be a finite set of positive integers such that for all different nonempty subsets B, C of A we have*

$$\sum_{x \in B} x \neq \sum_{x \in C} x.$$

Prove that

$$\sum_{a \in A} \frac{1}{a} < 2.$$

Solution. Let us order the elements $a_1 < \cdots < a_n$ of A and let us fix $k \in \{1, 2, \ldots, n\}$. By assumption two nonempty subsets of $\{a_1, \ldots, a_k\}$ have different sums. There are $2^k - 1$ such subsets and $a_1 + \cdots + a_k$ is the maximal sum, thus necessarily

$$a_1 + \cdots + a_k \geq 2^k - 1 = 1 + 2 + \cdots + 2^{k-1}.$$

Letting $b_k = 2^{k-1}$, we obtain

$$a_1 + \cdots + a_k \geq b_1 + \cdots + b_k$$

for $1 \leq k \leq n$. On the other hand, it is clear that $a_1 b_1 < \cdots < a_n b_n$, hence we can apply Example 4.4 and obtain

$$\frac{1}{a_1} + \cdots + \frac{1}{a_n} \leq \frac{1}{b_1} + \cdots + \frac{1}{b_n} = 1 + \frac{1}{2} + \cdots + \frac{1}{2^{n-1}} = 2\left(1 - \frac{1}{2^n}\right) < 2,$$

as needed.

Example 4.15. (USAMO 1982) *Prove that for all $x \geq 0$ and all $n \geq 1$ we have*

$$\lfloor nx \rfloor \geq \frac{\lfloor x \rfloor}{1} + \frac{\lfloor 2x \rfloor}{2} + \cdots + \frac{\lfloor nx \rfloor}{n}.$$

Solution. We will prove the result by induction, the case $n = 1$ being clear. Assume that the result holds up to $n - 1$ and let us prove it for n. Fix a nonnegative real number x and define

$$a_k = \frac{\lfloor x \rfloor}{1} + \frac{\lfloor 2x \rfloor}{2} + \cdots + \frac{\lfloor kx \rfloor}{k}.$$

We are asked to prove that $a_n \leq \lfloor nx \rfloor$. By Abel summation, we obtain

$$\sum_{k=1}^{n} \lfloor kx \rfloor = \sum_{k=1}^{n} \frac{\lfloor kx \rfloor}{k} \cdot k = na_n - \sum_{k=1}^{n-1} a_k.$$

By the inductive hypothesis we have $a_k \leq \lfloor kx \rfloor$ for $1 \leq k \leq n-1$. Thus

$$na_n \leq \sum_{k=1}^{n} \lfloor kx \rfloor + \sum_{k=1}^{n-1} \lfloor kx \rfloor = \lfloor nx \rfloor + \sum_{k=1}^{n-1} (\lfloor kx \rfloor + \lfloor (n-k)x \rfloor).$$

On the other hand, for all real numbers x, y we have

$$\lfloor x \rfloor + \lfloor y \rfloor \leq \lfloor x + y \rfloor,$$

as this reduces to the obvious inequality

$$\lfloor \{x\} + \{y\} \rfloor \geq 0,$$

where $\{x\} = x - \lfloor x \rfloor \geq 0$ and $\{y\} = y - \lfloor y \rfloor \geq 0$ are the fractional parts of x and y. Thus we obtain

$$na_n \leq \lfloor nx \rfloor + (n-1)\lfloor nx \rfloor = n\lfloor nx \rfloor,$$

which yields the desired result.

Chapter 5

Mathematical Induction

Mathematical induction can be very useful for proving statements (like identities or inequalities) when they depend on some positive integer. A proof by mathematical induction has two important parts. The first is the the base case: showing that the statement is true in some initial case. The second is the inductive step: checking that if the statement is true for one case, then it is also true for the immediately following case.

Example 5.1. *Prove that, for any positive integer n,*

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}.$$

Solution. This is a quite simple exercise. We have already seen a proof in the introduction, and now we give it as an example for the inductive method (and for the beginners). If $P(n)$ denotes the statement of the problem, we have

$$P(1):\ 1^2 = \frac{1 \cdot 2 \cdot 3}{6},$$

which is clearly true. We still need to show that $P(n) \Rightarrow P(n+1)$; that is, we need to get from

$$\sum_{k=1}^{n} k^2 = 1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

that

$$\sum_{k=1}^{n+1} k^2 = 1^2 + 2^2 + \cdots + n^2 + (n+1)^2 = \frac{(n+1)(n+2)(2n+3)}{6}.$$

Indeed, we have

$$\begin{aligned}
\sum_{k+1}^{n+1} k^2 = \sum_{k+1}^{n} k^2 + (n+1)^2 &= \frac{n(n+1)(2n+1)}{6} + (n+1)^2 \\
&= \frac{(n+1)(n(2n+1) + 6(n+1))}{6} \\
&= \frac{(n+1)(2n^2 + 7n + 6)}{6} \\
&= \frac{(n+1)(n+2)(2n+3)}{6}.
\end{aligned}$$

Note that knowing where we want to arrive is always helpful; for example, it helps for the last factorization.

Example 5.2. *A formula exists for the alternating sum of the first n squares. Prove that*

$$\sum_{k=1}^{n} (-1)^{k-1} k^2 = (-1)^{n-1} \frac{n(n+1)}{2}.$$

Solution. To verify this for $n = 1$ is easy, hence we prove that if the equality is true for some positive integer n, then it holds for $n + 1$, too. We have

$$\begin{aligned}
\sum_{k=1}^{n+1} (-1)^{k-1} k^2 = \sum_{k=1}^{n} (-1)^{k-1} k^2 + (-1)^n (n+1)^2 \\
&= (-1)^{n-1} \frac{n(n+1)}{2} + (-1)^n (n+1)^2 \\
&= (-1)^n (n+1) \left(-\frac{n}{2} + n + 1 \right) \\
&= (-1)^n \frac{(n+1)(n+2)}{2},
\end{aligned}$$

as desired. Can you telescope this sum?

Example 5.3. *Prove that*

$$\sum_{k=1}^{n}\left\lfloor\frac{k}{2}\right\rfloor=\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n+1}{2}\right\rfloor,$$

where $\lfloor x\rfloor$ represents the integer part of the real number x.

Solution. For $n=1$ both sides are equal to 0. If the equality is true for n, then it is true for $n+1$, too, since

$$\sum_{k=1}^{n+1}\left\lfloor\frac{k}{2}\right\rfloor=\sum_{k=1}^{n}\left\lfloor\frac{k}{2}\right\rfloor+\left\lfloor\frac{n+1}{2}\right\rfloor=\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n+1}{2}\right\rfloor+\left\lfloor\frac{n+1}{2}\right\rfloor$$
$$=\left\lfloor\frac{n+1}{2}\right\rfloor\left(1+\left\lfloor\frac{n}{2}\right\rfloor\right)=\left\lfloor\frac{n+1}{2}\right\rfloor\left\lfloor\frac{n+2}{2}\right\rfloor.$$

We used, for the last equality, the property $\lfloor x\rfloor+p=\lfloor x+p\rfloor$, for any real number x, and for any integer p.
Did you see the telescope? Basically we have

$$\sum_{k=1}^{n}\left\lfloor\frac{k}{2}\right\rfloor=\sum_{k=1}^{n}\left(\left\lfloor\frac{k}{2}\right\rfloor\left\lfloor\frac{k+1}{2}\right\rfloor-\left\lfloor\frac{k-1}{2}\right\rfloor\left\lfloor\frac{k}{2}\right\rfloor\right)=\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n+1}{2}\right\rfloor.$$

We invite you to prove in a similar manner that

$$\sum_{k=1}^{n}\left\lfloor\frac{k}{4}\right\rfloor=\left\lfloor\frac{n-1}{4}\right\rfloor\left\lfloor\frac{n+1}{4}\right\rfloor+\left\lfloor\frac{n}{4}\right\rfloor\left\lfloor\frac{n+2}{4}\right\rfloor$$

for any positive integer n.

Example 5.4. *Prove that for a prime number p, and any integers $x_1,x_2,\ldots,x_n$, we have*

$$(x_1+x_2+\cdots+x_n)^p\equiv x_1^p+x_2^p+\cdots+x_n^p\pmod{p}.$$

Solution. First we have

$$(x_1+x_2)^p=x_1^p+x_2^p+\sum_{j=1}^{p-1}\binom{p}{j}x_1^{p-j}x_2^j\equiv x_1^p+x_2^p\pmod{p},$$

because every binomial coefficient $\binom{p}{j}$ with $1 \leq j \leq p-1$ is divisible by p. Indeed,

$$\binom{p}{j} = \frac{p(p-1)\cdots(p-j+1)}{j!}$$

and we see that the factor p from the numerator cannot be cancelled, whence the conclusion follows. Thus we have our result proved for $n = 2$ (while for $n = 1$ it is obvious – no proof needed). Assuming it to be true for n arbitrary integers, we prove it for $n+1$ integers:

$$\begin{aligned}(x_1 + x_2 + \cdots + x_n + x_{n+1})^p &\equiv (x_1 + x_2 + \cdots + x_n)^p + x_{n+1}^p \\ &\equiv x_1^p + x_2^p + \cdots + x_n^p + x_{n+1}^p \pmod{p}\end{aligned}$$

by using successively the cases of two and n numbers.
Some remarks can be made about this (not very complicated, although useful) result. First, the same type of induction is often used in mathematics. For instance, one can prove by the same approach the generalized triangle inequality

$$|z_1 + z_2 + \cdots + z_n| \leq |z_1| + |z_2| + \cdots + |z_n|,$$

for complex numbers $z_1, z_2, \ldots, z_n$ (after proving that it holds for $n = 2$). Or, you can show that the determinant of the product of a number of matrices equals the product of the determinants of those matrices (once you proved that this true for two matrices). Or, somehow similarly, the reader can generalize the first congruence to

$$(x+y)^{p^j} \equiv x^{p^j} + y^{p^j} \pmod{p},$$

for integers x and y, and positive integer j, and so on.
Second, note that Fermat's "little" theorem (stating that $n^p - n$ is divisible by p for prime p and integer n) follows immediately from this result. It is enough to consider the above congruence for $x_1 = x_2 = \cdots = x_n = 1$, and we get

$$n^p \equiv n \pmod{p},$$

for each positive integer n. The same congruence is actually true for any integer n (think how) and is very useful in number theory. Also, if n and p are relatively prime (or, equivalently, if p does not divide n) we have

$$n^{p-1} \equiv 1 \pmod{p}.$$

Example 5.5. *For a positive integer $n \geq 2$ prove that any positive integer m with $1 \leq m \leq n^2$, $m \neq 2$ and $m \neq n^2 - 2$ can be expressed as the sum of some different numbers from the set $\{1, 3, \ldots, 2n-1\}$ (consisting of the first n odd positive integers). Also prove that $n^2 - 2$ does not have this property.*

Solution. We use induction; for $n = 2$ both results are obvious: we have $1 = 1$, $3 = 3$, $4 = 1 + 3$ and 2 cannot be written as the sum of some different numbers from the set $\{1, 3\}$. Also, for $n = 3$ we have $1 = 1$, $3 = 3$, $4 = 3 + 1$, $5 = 5$, $6 = 5 + 1$, $8 = 5 + 3$, $9 = 5 + 3 + 1$, while 2 and $7 = 3^2 - 2$ cannot be expressed in the required manner.

Now, the checking being made for $n = 2$ and $n = 3$, we assume that $n \geq 4$ and that any number from 1 to $(n-1)^2$, except 2 and $(n-1)^2 - 2$ can be expressed as a sum of some distinct numbers from the set $\{1, 3, \ldots, 2n-3\}$. Consider any $1 \leq m \leq n^2$, $m \neq 2$, $m \neq n^2 - 2$.

First, if $m \leq 2n - 1$ we have either the representation $m = m$ (when m is odd), or $m = (m-1) + 1$ (if m is even but not 2). Notice that we assumed $n \geq 4$, which ensures $n^2 - 2 > 2n - 1$.

In the second case, suppose that $2n \leq m \leq n^2$ and $m \neq n^2 - 2$. We then have $1 \leq m - (2n-1) \leq (n-1)^2$ and $m - (2n-1) \neq (n-1)^2 - 2$. If, further, $m \neq 2n + 1$ we also have $m - (2n-1) \neq 2$ and all the conditions are fulfilled to apply the inductive hypothesis in order to infer that $m - (2n-1)$ is the sum of a few different odd numbers from 1 to $2n - 3$. Then, of course, m will be the sum of these numbers and $2n - 1$. Finally, for $m = 2n + 1$ we have $2n + 1 = (2n-3) + 1 + 3$; again by the assumption $n > 3$ this is a good expression of n, because $2n - 3 > 3$.

The second statement is obvious. If a subset of $\{1, 3, \ldots, 2n-1\}$ has sum $n^2 - 2$, then the complementary subset would have sum 2, which clearly cannot occur. This is problem 1786 from *Mathematics Magazine*. The interested reader can find a slightly different solution by David Nacin in the same *Magazine* from February 2008.

Example 5.6. *For positive integers k and n and complex numbers $a_1, a_2, \ldots, a_n$ we define*

$$S(a_1, \ldots, a_n; k) = \sum_{\emptyset \neq S \subseteq \{1, \ldots, n\}} (-1)^{n-|S|} \left(\sum_{i \in S} a_i \right)^k$$

$$= \sum_{j=0}^{n-1} (-1)^j \sum_{1 \le i_1 < \cdots < i_{n-j} \le n} (a_{i_1} + \cdots + a_{i_{n-j}})^k.$$

Then we have

$$S(a_1, \ldots, a_n; k) = \begin{cases} 0, & k < n \\ n! a_1 a_2 \cdots a_n, & k = n. \end{cases}$$

Solution. For $n = 1$ this is clear, for $n = 2$ and $k = 1$ the identity is

$$(a_1 + a_2) - (a_1 + a_2) = 0,$$

while for $n = 2$ and $k = 2$ it says that

$$(a_1 + a_2)^2 - (a_1^2 + a_2^2) = 2a_1 a_2.$$

We assume the result to be true for n numbers and we prove it for $n + 1$ numbers.
We have

$$\begin{aligned} S(a_1, \ldots, a_{n+1}; k) &= \sum_{\emptyset \ne S \subseteq \{1, \ldots, n+1\}} (-1)^{n+1-|S|} \left(\sum_{i \in S} a_i \right)^k \\ &= \sum_{\emptyset \ne S \subseteq \{1, \ldots, n\}} (-1)^{n+1-|S|} \left(\sum_{i \in S} a_i \right)^k \\ &\quad + \sum_{S \subseteq \{1, \ldots, n\}} (-1)^{n-|S|} \left(\sum_{i \in S} a_i + a_{n+1} \right)^k \end{aligned}$$

by splitting the sum into two sums, the first corresponding to subsets of $\{1, \ldots, n+1\}$ that do not contain $n+1$, and the second sum having terms that correspond to subsets of the form $S \cup \{n+1\}$ with $S \subseteq \{1, \ldots, n\}$ (thus to subsets that contain $n + 1$). In the second sum there is one term corresponding to the empty set (which we isolate), and for every other term we use

the binomial formula; then we change the order of summation. Thus we get

$$\begin{aligned}
S(a_1,\ldots,a_{n+1};k) &= \sum_{\emptyset\neq S\subseteq\{1,\ldots,n\}} (-1)^{n+1-|S|}\left(\sum_{i\in S} a_i\right)^k \\
&\quad + \sum_{\emptyset\neq S\subseteq\{1,\ldots,n\}} (-1)^{n-|S|}\sum_{j=0}^{k}\binom{k}{j}a_{n+1}^{k-j}\left(\sum_{i\in S} a_i\right)^j + (-1)^n a_{n+1}^k \\
&= \sum_{\emptyset\neq S\subseteq\{1,\ldots,n\}} (-1)^{n+1-|S|}\left(\sum_{i\in S} a_i\right)^k \\
&\quad + \sum_{j=0}^{k}\binom{k}{j}a_{n+1}^{k-j}\sum_{\emptyset\neq S\subseteq\{1,\ldots,n\}} (-1)^{n-|S|}\left(\sum_{i\in S} a_i\right)^j + (-1)^n a_{n+1}^k.
\end{aligned}$$

Now we note that the first sum and the terms corresponding to $j = k$ in the second sum cancel each other; also the terms corresponding to $j = 0$ in the second sum and the isolated term $(-1)^n a_{n+1}^k$ together form the expression

$$a_{n+1}^k \sum_{S\subseteq\{1,\ldots,n\}} (-1)^{n-|S|} = a_{n+1}^k \sum_{r=0}^{n}(-1)^r\binom{n}{r} = 0,$$

hence we only remain with

$$\begin{aligned}
S(a_1,\ldots,a_{n+1};k) &= \sum_{j=1}^{k-1}\binom{k}{j}a_{n+1}^{k-j}\sum_{\emptyset\neq S\subseteq\{1,\ldots,n\}} (-1)^{n-|S|}\left(\sum_{i\in S} a_i\right)^j \\
&= \sum_{j=1}^{k-1}\binom{k}{j}a_{n+1}^{k-j}S(a_1,\ldots,a_n;j).
\end{aligned}$$

If $k < n+1$, then every term in this sum has $j < n$ and hence vanishes by the inductive hypothesis. Thus we get a sum of 0. If $k = n+1$, then the only non-zero term is the $j = n$ term, which equals

$$\binom{n+1}{n}a_{n+1}n!a_1\cdots a_n = (n+1)!a_1\cdots a_{n+1},$$

as desired.

Example 5.7. *Let* $a_0 = a_1 = 1$ *and*

$$a_{n+1} = 1 + \frac{a_1^2}{a_0} + \cdots + \frac{a_n^2}{a_{n-1}}$$

for $n \geq 1$. *Find* a_n *in closed form.*

Solution. We see, by using the recurrence relation, that $a_2 = 2$, $a_3 = 6$, and $a_4 = 24$. Therefore we have good reasons to guess that $a_n = n!$ for all positive integers n, and we prove this statement by inducting on n. We are left with proving that, if $a_k = k!$ for every $k \leq n$, then $a_{n+1} = (n+1)!$, too. Note that the recurrence relation also gives us

$$a_n = 1 + \frac{a_1^2}{a_0} + \cdots + \frac{a_{n-1}^2}{a_{n-2}},$$

consequently

$$a_{n+1} = a_n + \frac{a_n^2}{a_{n-1}},$$

for every $n \geq 1$. By the assumption we made we have

$$a_{n+1} = n! + \frac{(n!)^2}{(n-1)!} = n!\left(1 + \frac{n!}{(n-1)!}\right) = n!(n+1) = (n+1)!$$

and our claim is proved: we have $a_n = n!$ for every natural number n.

Example 5.8. *Let* $x_1 = -2$, $x_2 = -1$ *and*

$$x_{n+1} = \sqrt[3]{n(x_n^2 + 1) + 2x_{n-1}}$$

for $n \geq 2$. *Find* $x_1 + x_2 + \cdots + x_{2009}$.

Solution. One sees that $x_3 = 0$ and $x_4 = 1$, thus one may guess that $x_n = n-3$ holds for all positive integers n. We prove this by inducting on n; assuming that the result $x_k = k-3$ is true for all $k \leq n$ (in particular $x_n = n-3$ and $x_{n-1} = n-4$), we will have, by the given recurrence formula,

$$\begin{aligned} x_{n+1} &= \sqrt[3]{n((n-3)^2 + 1) + 2(n-4)} \\ &= \sqrt[3]{n^3 - 6n^2 + 12n - 8} \\ &= n - 2 = (n+1) - 3, \end{aligned}$$

completing the proof.
In this situation $x_1 + x_2 + \cdots + x_{2009} = 2009 \cdot 1002 = 2013018$ follows quickly, doesn't it?

Example 5.9. *Find the positive real numbers $x_1, x_2, \ldots$ that satisfy*

$$x_1^3 + x_2^3 + \cdots + x_n^3 = (x_1 + x_2 + \cdots + x_n)^2$$

for any positive integer n.

Solution. We have $x_1^3 = x_1^2$ (for $n = 1$) and $x_1 > 0$ yielding $x_1 = 1$. Then, for $n = 2$, we get

$$1 + x_2^3 = (1 + x_2)^2 \Leftrightarrow x_2^3 - x_2^2 - 2x_2 = 0 \Leftrightarrow x_2(x_2 + 1)(x_2 - 2) = 0,$$

hence (being positive) $x_2 = 2$. These results, and the well-known identity

$$1^3 + 2^3 + \cdots + n^3 = (1 + 2 + \cdots + n)^2$$

are good reasons to believe that $x_k = k$ for all positive integers k. Supposing that this is true for every $k \in \{1, 2, \ldots, n\}$ we will have (by hypothesis)

$$\begin{aligned}
&1^3 + 2^3 + \cdots + n^3 + x_{n+1}^3 = (1 + 2 + \cdots + n + x_{n+1})^2 \\
&\Leftrightarrow \left(\frac{n(n+1)}{2}\right)^2 + x_{n+1}^3 = \left(\frac{n(n+1)}{2} + x_{n+1}\right)^2 \\
&\Leftrightarrow x_{n+1}^3 - x_{n+1}^2 - n(n+1)x_{n+1} = 0 \\
&\Leftrightarrow x_{n+1}(x_{n+1} + n)(x_{n+1} - (n+1)) = 0,
\end{aligned}$$

therefore $x_{n+1} = n + 1$ follows, finishing the inductive proof.

Example 5.10. *Let $a_1, a_2, \ldots, a_n$ be distinct positive integers. Prove that*

$$a_1^2 + a_2^2 + \cdots + a_n^2 \geq \frac{2n+1}{3}(a_1 + a_2 + \cdots + a_n).$$

Solution. The statement is clearly true for $n = 1$ (it becomes $a_1^3 \geq a_1^2$), so let us assume it holds for any n distinct positive integers, and prove it for $n + 1$. Thus, we want to show that

$$a_1^2 + a_2^2 + \cdots + a_n^2 + a_{n+1}^2 \geq \frac{2n+3}{3}(a_1 + a_2 + \cdots + a_n + a_{n+1})$$

is true for any $n+1$ distinct positive integers $a_1, a_2, \ldots, a_{n+1}$. Due to the symmetry we can assume that $a_1 < a_2 < \cdots < a_n < a_{n+1}$.
We have, by the induction hypothesis, that

$$a_1^2 + a_2^2 + \cdots + a_n^2 \geq \frac{2n+1}{3}(a_1 + a_2 + \cdots + a_n).$$

If we could prove that

$$a_{n+1}^2 \geq \frac{2}{3}(a_1 + a_2 + \cdots + a_n) + \frac{2n+3}{3}a_{n+1},$$

adding these inequalities would yield the desired conclusion.
Thus, it remains to show that

$$a_{n+1}^2 \geq \frac{2}{3}(a_1 + a_2 + \cdots + a_n) + \frac{2n+3}{3}a_{n+1},$$

for any positive integers $a_1 < a_2 < \cdots < a_n < a_{n+1}$.
Because they are integers, the above inequalities actually say that

$$a_{n+1} \geq a_n + 1, \ a_{n+1} \geq a_{n-1} + 2,$$

and so on, until $a_{n+1} \geq a_1 + n$. Consequently,

$$\begin{aligned} &\frac{2}{3}(a_1 + a_2 + \cdots + a_n) + \frac{2n+3}{3}a_{n+1} \\ \leq\ &\frac{2}{3}(a_{n+1} - n + a_{n+1} - (n-1) + \cdots + a_{n+1} - 1) + \frac{2n+3}{3}a_{n+1} \\ =\ &\frac{2}{3}\left(na_{n+1} - \frac{n(n+1)}{2}\right) + \frac{2n+3}{3}a_{n+1}. \end{aligned}$$

And now we have

$$\frac{2}{3}\left(na_{n+1} - \frac{n(n+1)}{2}\right) + \frac{2n+3}{3}a_{n+1} \leq a_{n+1}^2$$

because it is equivalent to

$$(a_{n+1} - (n+1))\left(a_{n+1} - \frac{n}{3}\right) \geq 0,$$

which is true, since $a_{n+1} \geq a_1 + n \geq n + 1$. Our proof is done.
Try to prove that

$$a_1^3 + a_2^3 + \cdots + a_n^3 \geq \frac{n(n+1)}{2}(a_1 + a_2 + \cdots + a_n)$$

for any distinct positive integers $a_1, a_2, \ldots, a_n$.

Example 5.11. *Prove that the inequality*

$$x^n - nx + n - 1 \geq 0$$

holds for any positive real number x and any positive integer $n \geq 1$.

Solution. We show that

$$(1 + a_1)(1 + a_2) \cdots (1 + a_n) \geq 1 + a_1 + a_2 + \cdots + a_n$$

for any real numbers $a_1, a_2, \ldots, a_n$ all greater than -1 and all having the same sign (possibly, some of them are 0).
The base case $n = 1$ is obvious. If we have

$$(1 + a_1)(1 + a_2) \cdots (1 + a_n) \geq 1 + a_1 + a_2 + \cdots + a_n$$

we can multiply in both sides by $1 + a_{n+1}$ and get

$$(1 + a_1)(1 + a_2) \cdots (1 + a_n)(1 + a_{n+1}) \geq (1 + a_1 + a_2 + \cdots + a_n)(1 + a_{n+1})$$

$$= 1 + a_1 + a_2 + \cdots + a_n + a_{n+1} + a_{n+1}(a_1 + a_2 + \cdots + a_n).$$

Since all numbers $a_1, a_2, \ldots, a_{n+1}$ have the same sign, we have

$$a_{n+1}(a_1 + a_2 + \cdots + a_n) \geq 0$$

and the conclusion follows for $n + 1$ numbers.
Now let $x > 0$ and consider $a_1 = a_2 = \cdots = a_n = x - 1$ which satisfy the conditions for the above inequality. And the inequality reads

$$(1 + x - 1)^n \geq 1 + n(x - 1) \Leftrightarrow x^n - nx + n - 1 \geq 0,$$

exactly as we wanted to prove.
This is one of the many instances of Bernoulli's inequality. It can also be obtained by using the AM-GM inequality in the form

$$x^n + \underbrace{1+1+\cdots+1}_{n-1 \text{ times}} \geq n\sqrt[n]{x^n \cdot \underbrace{1\cdot 1\cdot \ldots \cdot 1}_{n-1 \text{ times}}} = nx.$$

Thus we see that the equality case holds if and only if $x = 1$.
It also holds in the form

$$x^r - rx + r - 1 \geq 0$$

for positive x and $r \geq 1$ (not necessarily an integer). If we take

$$r = \frac{n+1}{n} \quad \text{and} \quad x = 1 + \frac{1}{n+1}$$

we get one interesting application of Bernoulli's inequality (in this more general form), namely we get

$$\left(1+\frac{1}{n+1}\right)^{\frac{n+1}{n}} - \frac{n+1}{n}\left(1+\frac{1}{n+1}\right) + \frac{n+1}{n} - 1 > 0$$

$$\Leftrightarrow \left(1+\frac{1}{n+1}\right)^{\frac{n+1}{n}} > 1 + \frac{1}{n} \Leftrightarrow \left(1+\frac{1}{n+1}\right)^{n+1} > \left(1+\frac{1}{n}\right)^n.$$

That is, Bernoulli's inequality implies the monotonicity of the sequence with general term

$$\left(1+\frac{1}{n}\right)^n,$$

which defines the number e (as being its limit).

Example 5.12. *Prove that*

$$A_n = \sum_{k=0}^{n-1} r^k \cos kt = \frac{1 - r\cos t - r^n \cos nt + r^{n+1}\cos(n-1)t}{1 - 2r\cos t + r^2}$$

and

$$B_n = \sum_{k=0}^{n-1} r^k \sin kt = \frac{r\sin t - r^n \sin nt + r^{n+1}\sin(n-1)t}{1 - 2r\cos t + r^2}$$

for any real numbers r and t with $1-2r\cos t+r^2\neq 0$ and any positive integer n.

Solution. We already have shown that these equalities hold by using de Moivre's formula. Now we prove by induction the formula for A_n and leave B_n to the careful reader.
For $n=1$ we have to see that

$$A_1 = 1 = \frac{1-r\cos t - r\cos t + r^2}{1-2r\cos t + r^2},$$

which is clearly true. Then we have (supposing that the formula for A_n holds)

$$A_{n+1} = A_n + r^n\cos nt = \frac{1-r\cos t - r^n\cos nt + r^{n+1}\cos(n-1)t}{1-2r\cos t+r^2} + r^n\cos nt$$

$$= \frac{1-r\cos t - r^n\cos nt + r^{n+1}\cos(n-1)t + r^n\cos nt - 2r^{n+1}\cos t\cos nt + r^{n+2}\cos nt}{1-2r\cos t + r^2}$$

$$= \frac{1-r\cos t - r^{n+1}\cos(n+1)t + r^{n+2}\cos nt}{1-2r\cos t+r^2},$$

because

$$r^{n+1}\cos(n-1)t - 2r^{n+1}\cos t\cos nt$$

$$= r^{n+1}(\cos t\cos nt + \sin t\sin nt - 2\cos t\cos nt) = -r^{n+1}\cos(n+1)t.$$

Thus the formula for A_{n+1} holds and we are done.

Example 5.13. *For each positive integer n and each real number x prove the following inequality*

$$|\cos x| + |\cos 2x| + |\cos 4x| + \cdots + |\cos 2^n x| \geq \frac{n}{2\sqrt{2}}.$$

Solution. First we observe that the inequality

$$|a| + |2a^2-1| \geq \frac{1}{\sqrt{2}}$$

holds for any real number a.

Indeed, this clearly holds for $|a| \geq 1/\sqrt{2}$, while for $|a| \leq 1/\sqrt{2}$ it can be written in the form

$$|a| + 1 - 2a^2 \geq \frac{1}{\sqrt{2}}.$$

For example, when $a \in \left[0, 1/\sqrt{2}\right]$ we have to prove that

$$f(a) = 1 + a - 2a^2 \geq \frac{1}{\sqrt{2}}.$$

But f is a quadratic function with maximum attained at $a = 1/4 \in \left[0, 1/\sqrt{2}\right]$ (the maximum is $9/8$, but this is not important in this matter). Therefore, the minimum value of the function in the interval $\left[0, 1/\sqrt{2}\right]$ is

$$\min\left\{f(0), f\left(\frac{1}{\sqrt{2}}\right)\right\} = \frac{1}{\sqrt{2}}.$$

Thus

$$|a| + |2a^2 - 1| = f(a) \geq \frac{1}{\sqrt{2}}$$

follows for $a \in \left[0, 1/\sqrt{2}\right]$, too. The reader will surely be able to prove the inequality in the case $a \in \left[-1/\sqrt{2}, 0\right]$ by a similar reasoning (or just by noting the parity of the function $a \mapsto |a| + |2a^2 - 1|$). In particular, we have

$$|\cos t| + |\cos 2t| = |\cos t| + |2\cos^2 t - 1| \geq \frac{1}{\sqrt{2}}$$

for every real t, implying that $P(1)$ is true (it is weaker than the above inequality), if we denote by $P(n)$ the statement of the problem.
For $P(2)$ we have

$$|\cos x| + |\cos 2x| + |\cos 4x| \geq |\cos x| + |\cos 2x| \geq \frac{1}{\sqrt{2}} = \frac{2}{2\sqrt{2}},$$

according to the same inequality proved above.
And now we show that $P(n-2)$ implies $P(n)$. Indeed

$$\sum_{k=0}^{n} |\cos 2^k x| = \sum_{k=0}^{n-2} |\cos 2^k x| + |\cos 2^{n-1} x| + |\cos 2^n x| \geq \frac{n-2}{2\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{n}{2\sqrt{2}}$$

and the inductive proof is done. Note that the following proof is also available (also based on $|\cos t| + |\cos 2t| \geq 1/\sqrt{2}$). Adding the inequalities

$$|\cos 2^{k-1}x| + |\cos 2^k x| \geq \frac{1}{\sqrt{2}},\ k = 1, 2, \dots, n$$

(together with $|\cos x| \geq |\cos x|$ and $|\cos 2^n x| \geq |\cos 2^n x|$) yields

$$2\sum_{k=0}^{n} |\cos 2^k x| \geq \frac{n}{\sqrt{2}} + |\cos x| + |\cos 2^n x|$$

and since the sum of absolute values from the right side is nonnegative, we obtain the desired inequality

$$2\sum_{k=0}^{n} |\cos 2^k x| \geq \frac{n}{\sqrt{2}}.$$

It is a matter of taste if we choose the proof that uses induction, or the one that avoids it. The most interesting fact about the two proofs above is that there is one simpler than each of them. We are sure that, carefully analyzing the cases $n = 1$ and $n = 2$, the reader will be able to find it. The lesson to be learned here is that a proposition depending on a positive integer variable need not be proven by induction.

Example 5.14. *Prove that Jackson's inequality:*

$$\sin x + \frac{\sin 2x}{2} + \cdots + \frac{\sin nx}{n} > 0$$

holds for any $x \in (0, \pi)$.

Solution. Let

$$S_n(x) = \sin x + \frac{\sin 2x}{2} + \cdots + \frac{\sin nx}{n}.$$

For $n = 1$ we clearly have

$$S_1(x) = \sin x > 0$$

for all $x \in (0, \pi)$. Assume that $S_n(x) > 0$ for every $x \in (0, \pi)$ and let us prove that the same is true for $S_{n+1}(x)$. The derivative of S_{n+1} as a function of x is

$$S'_{n+1}(x) = \cos x + \cos 2x + \cdots + \cos(n+1)x = \frac{\sin \dfrac{(n+1)x}{2} \cos \dfrac{(n+2)x}{2}}{\sin \dfrac{x}{2}}$$

and has zeros (hence the critical points of S_{n+1})

$$x'_k = \frac{2k\pi}{n+1}$$

and

$$x''_k = \frac{(2k+1)\pi}{n+2}$$

for $0 < 2k < n+1$ (since we want x'_k and x''_k to be in the interval $(0, \pi)$). The extremum points of S_{n+1} are among these points and the endpoints, and we have, by using the inductive hypothesis,

$$S_{n+1}(x'_k) = S_n(x'_k) + \frac{\sin(n+1)x'_k}{n+1} = S_n(x'_k) > 0$$

and

$$\begin{aligned} S_{n+1}(x''_k) &= S_n(x''_k) + \frac{\sin(n+1)x''_k}{n+1} = S_n(x''_k) + \frac{\sin \dfrac{(n+1)(2k+1)\pi}{n+2}}{n+1} \\ &= S_n(x''_k) + \frac{\sin \dfrac{(2k+1)\pi}{n+2}}{n+1} = S_n(x''_k) + \frac{\sin x''_k}{n+1} > 0 \end{aligned}$$

because $x''_k \in (0, \pi)$. Also, the limits of S_{n+1} at the endpoints 0 and π of the interval are both equal to 0, therefore the conclusion follows. We invite the reader to prove that

$$\sum_{k=1}^{n} (-1)^{k+1} \frac{\sin kx}{k} > 0$$

for every $x \in (0, \pi)$ (not necessarily by induction, but rather as a consequence of the result that we have proved).

Example 5.15. *Prove that*

$$\sum_{k=1}^{n} \frac{1}{k}\binom{n}{k} = \sum_{k=1}^{n} \frac{2^k - 1}{k}.$$

Solution. Let S_n and T_n denote the left-hand sum and the right-hand side respectively. We have

$$\begin{aligned} S_{n+1} - S_n &= \frac{1}{n+1} + \sum_{k=1}^{n} \frac{1}{k}\left(\binom{n+1}{k} - \binom{n}{k}\right) = \frac{1}{n+1} + \sum_{k=1}^{n} \frac{1}{k}\binom{n}{k-1} \\ &= \frac{1}{n+1} + \frac{1}{n+1}\sum_{k=1}^{n}\binom{n+1}{k} = \frac{1}{n+1} + \frac{2^n - 2}{n+1} = \frac{2^n - 1}{n+1}. \end{aligned}$$

In the previous calculation, we used the recurrence formula of the binomial coefficients, then the formula

$$\frac{1}{k}\binom{n}{k-1} = \frac{n!}{k!(n+1-k)!} = \frac{1}{n+1}\binom{n+1}{k},$$

and finally the sum of the binomial coefficients from the $(n+1)$-st row of Pascal's triangle.
On the other hand, it is clear that

$$T_{n+1} - T_n = \frac{2^n - 1}{n+1}.$$

Now we have $S_1 = T_1$, and if $S_n = T_n$ we also get

$$S_{n+1} = (S_{n+1} - S_n) + S_n = (T_{n+1} - T_n) + T_n = T_{n+1}$$

and the identity follows for all positive integers n.

Example 5.16. (USA TST, 2000) *Let n be a positive integer. Prove that*

$$\binom{n}{0}^{-1} + \binom{n}{1}^{-1} + \cdots + \binom{n}{n}^{-1} = \frac{n+1}{2^{n+1}}\left(\frac{2}{1} + \frac{2^2}{2} + \cdots + \frac{2^{n+1}}{n+1}\right).$$

Solution. The proof is by induction on n. Denote the left hand side by a_n, and the right hand side by b_n. The base case is clear and we only need to show that a_n and b_n satisfy the same recurrence. A recurrence for b_n is simple:

$$b_n = \frac{n+1}{2n} b_{n-1} + 1.$$

So we only need to prove that

$$a_n = \frac{n+1}{2n} a_{n-1} + 1.$$

We have

$$\frac{n+1}{2n} \binom{n-1}{i}^{-1} = \frac{(n+1)i!(n-i-1)!}{2(n!)}.$$

To express the right hand side in terms of binomial coefficients of base n, we write $n+1$ as $(i+1)+(n-i)$ and conclude that

$$\begin{aligned} \frac{n+1}{2n} \binom{n-1}{i}^{-1} &= \frac{((i+1)+(n-i))i!(n-i-1)!}{2(n!)} \\ &= \frac{1}{2}\left(\binom{n}{i+1}^{-1} + \binom{n}{i}^{-1}\right). \end{aligned}$$

By summing these relations and using the fact that

$$\binom{n}{0} = \binom{n}{n} = 1,$$

we get

$$a_n = \frac{n+1}{2n} a_{n-1} + 1,$$

and we are done.

Chapter 6

Combinatorial Identities and Generating Functions

Combinatorial identities deal with binomial coefficients. The binomial coefficients $\binom{n}{k}$ are defined by

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \text{ if } 0 \le k \le n, \text{ and } \binom{n}{k} = 0, \text{ otherwise,}$$

where n is a nonnegative integer, while k is an integer. Remember that $0!$ is defined to be 1.

A very useful relation is the recurrence formula of the binomial coefficients:

$$\binom{n+1}{k+1} = \binom{n}{k} + \binom{n}{k+1},$$

which you can check applying the definition. The binomial coefficients can be found using the binomial theorem (Newton's formula)

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k.$$

Many useful formulas are easy consequences of the binomial theorem.

For example, the above formula with $a = b = 1$ yields

$$\sum_{k=0}^{n} \binom{n}{k} = \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = 2^n,$$

a relation that is useful in many combinatorial identities. The binomial coefficients appearing here are in the same (the $(n+1)$-st) row of Pascal's triangle. We also have (the generalized Newton's binomial formula)

$$(1+x)^\alpha = \binom{\alpha}{0} + \binom{\alpha}{1}x + \binom{\alpha}{2}x^2 + \cdots = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k$$

where the binomial coefficient is defined for an arbitrary real number α as

$$\binom{\alpha}{k} = \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!}.$$

In the particular case $\alpha = n$ (a nonnegative integer) we recapture the binomial formula with a finite development:

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \cdots + \binom{n}{n-1}x^{n-1} + \binom{n}{n}x^n + \binom{n}{n+1}x^{n+1} + \cdots,$$

because $\binom{n}{k} = 0$ for $k > n$. The function $(1+x)^n$ is called the generating function of the sequence $c_k = \binom{n}{k}$. For every sequence $(a_n)_{n\geq 0}$ we can associate an infinite series

$$F(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$$

called the (ordinary) generating function of the sequence. As an example let us take the sequence $a_i = 1$, then

$$F(x) = 1 + x + x^2 + \cdots + x^n + \cdots = \frac{1}{1-x}.$$

Thus

$$F(x) = \frac{1}{1-x}$$

is the generating function of the sequence $a_i = 1$. Let us list some of the most important sequences and functions corresponding to them: The sequence $a_k = 1/k$ corresponds to

$$-\ln(1-x) = x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \cdots,$$

from this result it immediately follows that

$$\ln 2 = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{i} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

The series corresponding to the sequence $a_k = 1/k!$ describes the universal constant e:

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

For the Fibonnaci sequence ($F_0 = F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$) the generating function is

$$\frac{1}{1-x-x^2} = 1 + x + 2x^2 + 3x^3 + 5x^4 + \cdots$$

and, of course, Newton's formula says that the sequence $a_k = \binom{\alpha}{k}$ corresponds to:

$$(1+x)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n.$$

Example 6.1. *Evaluate*

$$\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots$$

Solution. For $n \geq 1$ define

$$A = \binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k}$$

and

$$B = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \cdots = \sum_{k=1}^{\lfloor (n+1)/2 \rfloor} \binom{n}{2k-1}$$

we have

$$A + B = \sum_{j=0}^{n} \binom{n}{j} = (1+1)^n = 2^n,$$

$$A - B = \sum_{j=0}^{n} (-1)^j \binom{n}{j} = (1-1)^n = 0,$$

therefore $A = B = 2^{n-1}$. These are also fundamental identities satisfied by the binomial coefficients.

Example 6.2. *Prove that*

$$\sum_{k=1}^{n} k \binom{n}{k} = n \cdot 2^{n-1}.$$

Solution 1. Because

$$k\binom{n}{k} = k\frac{n!}{k!(n-k)!} = n\frac{(n-1)!}{(k-1)!(n-k)!} = n\binom{n-1}{k-1},$$

we get

$$\sum_{k=1}^{n} k\binom{n}{k} = \sum_{k=1}^{n} n\binom{n-1}{k-1} = n\sum_{j=0}^{n-1}\binom{n-1}{j} = n \cdot 2^{n-1},$$

according to the fundamental identity of the binomial coefficients.

Solution 2. We start with the binomial formula in the form

$$\sum_{k=0}^{n} \binom{n}{k} x^k = (1+x)^n,$$

valid for any real number x. Differentiating with respect to x yields

$$\sum_{k=1}^{n} k\binom{n}{k} x^{k-1} = n(1+x)^{n-1},$$

which is a more general identity – it is enough to take $x = 1$ in order to obtain the result stated by the problem.
Try to prove in a similar manner the identity

$$\sum_{k=2}^{n} k(k-1)\binom{n}{k} = n(n-1)\cdot 2^{n-2}$$

(use $k(k-1)\binom{n}{k} = n(n-1)\binom{n-2}{k-2}$ or differentiate twice), then infer

$$\sum_{k=1}^{n} k^2\binom{n}{k} = n(n+1)\cdot 2^{n-2},$$

for any positive integer n. Also prove that for $n \geq 3$

$$\sum_{k=1}^{n}(-1)^k k\binom{n}{k} = \sum_{k=1}^{n}(-1)^k k^2\binom{n}{k} = 0.$$

Example 6.3. *For nonnegative integers m and n show that*

$$\sum_{k=0}^{m}(-1)^k\binom{n}{k} = (-1)^m\binom{n-1}{m}.$$

Solution. The result is clear for $m = 0$. Assuming that it holds for some m, we will have

$$\begin{aligned}
\sum_{k=0}^{m+1}(-1)^k\binom{n}{k} &= \sum_{k=0}^{m}(-1)^k\binom{n}{k} + (-1)^{m+1}\binom{n}{m+1} \\
&= (-1)^m\binom{n-1}{m} + (-1)^{m+1}\binom{n}{m+1} \\
&= (-1)^{m+1}\left(-\binom{n-1}{m} + \binom{n}{m+1}\right) \\
&= (-1)^{m+1}\binom{n-1}{m+1},
\end{aligned}$$

as we needed to prove, according to the recurrence of the binomial coefficients. Note that the result from the right-hand side is 0 for $m \geq n \geq 1$. Actually, for $m \geq n \geq 1$ the identity is always the same, namely

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k} = 0,$$

which by the binomial theorem basically says $(1-1)^n = 0$. Also observe that a direct proof (by telescoping) can be obtained if we replace each $\binom{n}{k}$ with $\binom{n-1}{k} + \binom{n-1}{k-1}$, by the same recurrence relation for the binomial coefficients.

Example 6.4. *For positive integers m and n with $m \leq n$ prove that*

$$\sum_{k=m}^{n}\binom{k}{m}\binom{n}{k} = 2^{n-m}\binom{n}{m}.$$

Solution. We have

$$\begin{aligned}\binom{k}{m}\binom{n}{k} &= \frac{k!}{m!(k-m)!}\cdot\frac{n!}{k!(n-k)!}\\ &= \frac{n!}{m!(n-m)!}\cdot\frac{(n-m)!}{(n-k)!(k-m)!}\\ &= \binom{n}{m}\binom{n-m}{n-k},\end{aligned}$$

therefore

$$\sum_{k=m}^{n}\binom{k}{m}\binom{n}{k} = \sum_{k=m}^{n}\binom{n}{m}\binom{n-m}{n-k} = \binom{n}{m}\sum_{j=0}^{n-m}\binom{n-m}{j} = 2^{n-m}\binom{n}{m}.$$

We changed the index of summation with the substitution $j = n-k$, when k runs from m to n we see that j runs from $n-m$ to 0.

Example 6.5. *Evaluate*

$$S_n = \sum_{k=0}^{n}\frac{1}{2^k}\binom{n+k}{k}.$$

Solution. We have $S_0 = 1$, $S_1 = 2$, $S_2 = 4$ and $S_3 = 8$, therefore we have reasons to believe that $S_n = 2^n$ for every nonnegative integer n. We can prove this by finding a recurrence relation for S_n. Note that

$$\binom{n+1+k}{k} = \binom{n+k}{k} + \binom{n+k}{k-1} \text{ for } k \geq 1$$

and

$$\binom{n+1+0}{0} = \binom{n+0}{0} \text{ for } k = 0.$$

Thus

$$S_{n+1} = \sum_{k=0}^{n+1} \frac{1}{2^k}\binom{n+k}{k} + \sum_{k=1}^{n+1} \frac{1}{2^k}\binom{n+k}{k-1}$$

$$= \sum_{k=0}^{n} \frac{1}{2^k}\binom{n+k}{k} + \frac{1}{2^{n+1}}\binom{2n+1}{n+1} + \sum_{k=1}^{n+2} \frac{1}{2^k}\binom{n+k}{k-1} - \frac{1}{2^{n+2}}\binom{2n+2}{n+1}.$$

Since

$$\frac{1}{2}\binom{2n+2}{n+1} = \frac{1}{2}\frac{(2n+2)(2n+1)!}{(n+1)n!(n+1)!} = \binom{2n+1}{n+1}$$

the two terms extracted from the sums above cancel, and we get

$$\begin{aligned} S_{n+1} &= \sum_{k=0}^{n} \frac{1}{2^k}\binom{n+k}{k} + \sum_{k=1}^{n+2} \frac{1}{2^k}\binom{n+k}{k-1} \\ &= S_n + \frac{1}{2}\sum_{k=1}^{n+2} \frac{1}{2^{k-1}}\binom{(n+1)+(k-1)}{k-1} \\ &= S_n + \frac{1}{2}S_{n+1}, \end{aligned}$$

that is, $S_{n+1} = 2S_n$. Now an easy induction (based on this recurrence relation) shows that, indeed, $S_n = 2^n$ for all $n \geq 0$.

Example 6.6. *Prove that*

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \cdots + \binom{n}{n}^2 = \binom{2n}{n}.$$

Solution. Consider the function $f(x) = (1+x)^{2n}$. From the Binomial Theorem the coefficient of x^n in $f(x)$ is equal to $\binom{2n}{n}$. On the other hand, we have

$$(1+x)^{2n} = (1+x)^n(1+x)^n$$

$$= \left(\binom{n}{0} + \binom{n}{1}x + \cdots \binom{n}{n}x^n\right)\left(\binom{n}{0} + \binom{n}{1}x + \cdots \binom{n}{n}x^n\right).$$

If we perform standard multiplication, the coefficient of x^n will be equal to

$$\binom{n}{0}\binom{n}{n} + \binom{n}{1}\binom{n}{n-1} + \cdots + \binom{n}{n}\binom{n}{0}.$$

But $\binom{n}{i} = \binom{n}{n-i}$, thus we get

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \cdots + \binom{n}{n}^2 = \binom{2n}{n}.$$

Example 6.7. *Evaluate*

$$\binom{n}{0} + \binom{n}{4} + \binom{n}{8} + \cdots$$

Solution. Let us denote by A the sum from the problem statement, and let us consider also

$$B = \binom{n}{1} + \binom{n}{5} + \binom{n}{9} + \cdots,$$
$$C = \binom{n}{2} + \binom{n}{6} + \binom{n}{10} + \cdots,$$

and

$$D = \binom{n}{3} + \binom{n}{7} + \binom{n}{11} + \cdots.$$

Then for $n \geq 1$ we clearly have (by the binomial expansion)

$$A + B + C + D = \sum_{k=0}^{n} \binom{n}{k} = (1+1)^n = 2^n,$$

$$A - B + C - D = \sum_{k=0}^{n} (-1)^k \binom{n}{k} = (1-1)^n = 0,$$

$$A + iB - C - iD = \sum_{k=0}^{n} \binom{n}{k} i^k = (1+i)^n,$$

and

$$A - iB - C + iD = \sum_{k=0}^{n} \binom{n}{k} (-i)^k = (1-i)^n.$$

The system of four linear equations with unknowns A, B, C, and D can be easily solved (by adding and subtracting the first two and the last two equations, etc.), yielding

$$A = \frac{1}{2}\left(2^{n-1} + 2^{\frac{n}{2}} \cos \frac{n\pi}{4}\right).$$

One can also get

$$B = \frac{1}{2}\left(2^{n-1} + 2^{\frac{n}{2}} \sin \frac{n\pi}{4}\right), \quad C = \frac{1}{2}\left(2^{n-1} - 2^{\frac{n}{2}} \cos \frac{n\pi}{4}\right)$$

and

$$D = \frac{1}{2}\left(2^{n-1} - 2^{\frac{n}{2}} \sin \frac{n\pi}{4}\right).$$

Note that these formulae do not work for $n = 0$, because in that case $A - B + C - D$ equals 1, not 0.

Example 6.8. *Evaluate the sum*

$$\sum_{k=0}^{\lfloor \frac{n}{3} \rfloor} \binom{n}{3k}.$$

Solution. From the binomial theorem, this is equivalent to finding the sum of the coefficients of $f(x) = (1+x)^n$ for the powers of x whose exponents are divisible by 3. Let

$$\omega = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3},$$

a third root of unity. Then we have the following identity

$$1 + \omega^k + \omega^{2k} = \begin{cases} 3 & \text{if} \quad 3 \mid k \\ 0 & \text{otherwise} \end{cases}$$

The proof is simple, and amounts to taking cases on the value of k modulo 3. If $k \equiv 0 \pmod 3$, then $\omega^k = \omega^{2k} = 1$, and the above sum equals 3. If $k \equiv 1 \pmod 3$, then from $\omega^3 = 1$ we have $\omega^k = \omega$ and $\omega^{2k} = \omega^2$. The sum equals

$$1 + \omega + \omega^2 = \frac{\omega^3 - 1}{\omega - 1} = 0.$$

The case $k \equiv 2 \pmod 3$ is analogous.
Now, consider the quantity $\dfrac{f(1) + f(\omega) + f(\omega^2)}{3}$. By the above identity and the binomial theorem, it follows that

$$\begin{aligned}\sum_{k=0}^{\lfloor \frac{n}{3} \rfloor} \binom{n}{3k} &= \frac{f(1) + f(\omega) + f(\omega^2)}{3} \\ &= \frac{(1+1)^n + (1+\omega)^n + (1+\omega^2)^n}{3} \\ &= \frac{2^n + (-\omega)^n + (-\omega^2)^n}{3},\end{aligned}$$

where we have used the identity $1+\omega+\omega^2 = 0$ for the last equality. We invite the reader to check that the final result is

$$\sum_{k=0}^{\lfloor \frac{n}{3} \rfloor} \binom{n}{3k} = \frac{1}{3}\left(2^n + 2\cos\frac{n\pi}{3}\right),$$

by evaluating the powers of $-\omega$ and $-\omega^2$ with de Moivre's formula.

Example 6.9. *Find in closed form*

$$S_n = \sum_{k=0}^{n} (-1)^k (n-k)!(n+k)!.$$

Solution 1. Because

$$\binom{2n}{j} = \frac{(2n)!}{j!(2n-j)!} = \binom{2n}{2n-j}$$

our sum is easily transformed into a sum involving (reciprocals of) binomial coefficients (in two forms):

$$\frac{S_n}{(2n)!} = \sum_{k=0}^{n} \frac{(-1)^k}{\dbinom{2n}{n-k}} = \sum_{k=0}^{n} \frac{(-1)^k}{\dbinom{2n}{n+k}}.$$

Thus

$$2\frac{S_n}{(2n)!} = \sum_{k=0}^{n} \left(\frac{(-1)^k}{\dbinom{2n}{n-k}} + \frac{(-1)^k}{\dbinom{2n}{n+k}} \right) = \frac{1}{\dbinom{2n}{n}} + (-1)^n \sum_{j=0}^{2n} \frac{(-1)^j}{\dbinom{2n}{j}}.$$

Now the formula

$$\frac{1}{\dbinom{2n}{j}} = \frac{2n+1}{2n+2} \left(\frac{1}{\dbinom{2n+1}{j}} + \frac{1}{\dbinom{2n+1}{j+1}} \right)$$

can be easily checked by using standard computations (and we invite the reader to do that). So

$$\sum_{j=0}^{2n} \frac{(-1)^j}{\dbinom{2n}{j}} = \frac{2n+1}{2n+2} \sum_{j=0}^{2n} (-1)^j \left(\frac{1}{\dbinom{2n+1}{j}} + \frac{1}{\dbinom{2n+1}{j+1}} \right)$$

$$= \frac{2n+1}{2n+2} \sum_{j=0}^{2n} \left(\frac{(-1)^j}{\binom{2n+1}{j}} - \frac{(-1)^{j+1}}{\binom{2n+1}{j+1}} \right)$$

$$= \frac{2n+1}{2n+2} \left(\frac{(-1)^0}{\binom{2n+1}{0}} - \frac{(-1)^{2n+1}}{\binom{2n+1}{2n+1}} \right)$$

$$= \frac{2n+1}{n+1},$$

leading to

$$2\frac{S_n}{(2n)!} = \frac{1}{\binom{2n}{n}} + (-1)^n \cdot \frac{2n+1}{n+1}.$$

Finally we get

$$S_n = \frac{1}{2}(n!)^2 + (-1)^n \cdot \frac{(2n+1)!}{2(n+1)}.$$

It hardly seemed in the beginning that we would get a telescope for this sum, did it?

Solution 2. This is more involved, but showcases a useful trick. As in the first solution, we have

$$\frac{S_n}{(2n)!} = \sum_{k=0}^{n} \frac{(-1)^k}{\binom{2n}{n-k}}.$$

Now, the reciprocal of a binomial coefficient can be expressed by an integral with the following formula:

$$\frac{1}{\binom{m}{j}} = (m+1) \int_0^1 x^j (1-x)^{m-j} dx,$$

and consequently we have

$$\begin{aligned}\frac{S_n}{(2n)!} &= (2n+1)\sum_{k=0}^{n}(-1)^k\int_0^1 x^{n-k}(1-x)^{n+k}dx\\ &= (2n+1)\int_0^1 x^n(1-x)^n\sum_{k=0}^{n}\left(\frac{x-1}{x}\right)^k dx,\end{aligned}$$

where we arrived at the sum of a geometric progression:

$$\begin{aligned}\sum_{k=0}^{n}\left(\frac{x-1}{x}\right)^k &= \frac{\left(\dfrac{x-1}{x}\right)^{n+1}-1}{\frac{x-1}{x}-1} = \frac{x^{n+1}-(x-1)^{n+1}}{x^n}\\ &= \frac{x^{n+1}+(-1)^n(1-x)^{n+1}}{x^n}.\end{aligned}$$

We get further

$$\begin{aligned}\frac{S_n}{(2n)!} &= (2n+1)\int_0^1\left(x^{n+1}(1-x)^n+(-1)^n(1-x)^{2n+1}\right)dx\\ &= \frac{2n+1}{2n+2}\left(\frac{1}{\dbinom{2n+1}{n}}+(-1)^n\right),\end{aligned}$$

where, for $\displaystyle\int_0^1 x^{n+1}(1-x)^n dx$ we also used the formula from the beginning, with $m = 2n+1$ and $j = n+1$ (and we wrote $\binom{2n+1}{n}$ instead of $\binom{2n+1}{n+1}$). Finally, after multiplying by $(2n)!$ and a few more calculations, we obtain our result

$$S_n = \frac{n!}{2}(n! + (-1)^n(n+2)\cdots(2n+1)).$$

To prove the integral formula for the reciprocal of a binomial coefficient, notice that we have (integrating by parts)

$$I_j = \int_0^1 x^j(1-x)^{n-j}dx$$

$$= \frac{1}{j+1} x^{j+1}(1-x)^{n-j}\Big|_0^1 + \frac{n-j}{j+1}\int_0^1 x^{j+1}(1-x)^{n-j-1}dx,$$

thus

$$I_{j+1} = \frac{j+1}{n-j} I_j.$$

The desired formula follows by iterating this relation, until we arrive at

$$I_0 = \frac{1}{n+1},$$

which is easy to check.

Example 6.10. *Prove that*

$$\sum_{n=4}^{\infty}\left(\sum_{k=2}^{n-2}\binom{n}{k}^{-1}\right) = \frac{3}{2}.$$

Solution. We have

$$\frac{k}{k-1}\left(\frac{1}{\binom{n-1}{k-1}} - \frac{1}{\binom{n}{k-1}}\right) = \frac{k}{k-1}\cdot\frac{\binom{n}{k-1} - \binom{n-1}{k-1}}{\binom{n}{k-1}\binom{n-1}{k-1}}$$

$$= \frac{k}{k-1}\cdot\frac{\binom{n-1}{k-2}}{\binom{n}{k-1}\binom{n-1}{k-1}} = \frac{k}{k-1}\left(\frac{1}{\frac{n}{k-1}\binom{n-1}{k-1}}\right) = \frac{1}{\frac{n}{k}\binom{n-1}{k-1}} = \frac{1}{\binom{n}{k}}$$

therefore

$$\begin{aligned}
\sum_{n=4}^{\infty}\left(\sum_{k=2}^{n-2}\binom{n}{k}^{-1}\right) &= \sum_{k=2}^{\infty}\left(\sum_{n=k+2}^{\infty}\binom{n}{k}^{-1}\right)\\
&= \sum_{k=2}^{\infty}\left(\sum_{n=k+2}^{\infty}\frac{k}{k-1}\left(\binom{n-1}{k-1}^{-1} - \binom{n}{k-1}^{-1}\right)\right)\\
&= \sum_{k=2}^{\infty}\left(\frac{k}{k-1}\binom{k+1}{k-1}^{-1}\right) = \sum_{k=2}^{\infty}\left(\frac{k}{k-1}\binom{k+1}{2}^{-1}\right).
\end{aligned}$$

Finally, our sum is

$$\begin{aligned}\sum_{k=2}^{\infty}\left(\frac{k}{k-1}\binom{k+1}{2}^{-1}\right) &= \sum_{k=2}^{\infty}\left(\frac{k}{k-1}\cdot\frac{2}{(k+1)k}\right) = \sum_{k=2}^{\infty}\frac{2}{(k-1)(k+1)}\\ &= \sum_{i=0}^{\infty}\frac{2}{(2i+1)(2i+3)} + \sum_{i=0}^{\infty}\frac{2}{(2i+2)(2i+4)}\\ &= \sum_{i=0}^{\infty}\left(\frac{1}{2i+1}-\frac{1}{2(i+1)+1}\right) + \sum_{i=1}^{\infty}\left(\frac{1}{2i}-\frac{1}{2(i+1)}\right)\\ &= \frac{1}{1}+\frac{1}{2}=\frac{3}{2}.\end{aligned}$$

We can also evaluate the final sum directly as a telescope with step 2:

$$\begin{aligned}\sum_{k=2}^{\infty}\frac{2}{(k-1)(k+1)} &= \lim_{n\to\infty}\sum_{k=2}^{n}\frac{2}{(k-1)(k+1)} = \lim_{n\to\infty}\sum_{k=2}^{n}\left(\frac{1}{k-1}-\frac{1}{k+1}\right)\\ &= \lim_{n\to\infty}\left(1+\frac{1}{2}-\frac{1}{n}-\frac{1}{n+1}\right) = 1+\frac{1}{2}=\frac{3}{2}.\end{aligned}$$

Example 6.11. *Evaluate*

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^{m+n}}{m!n!}.$$

Solution. Observe the following fact

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^{m+n}}{m!n!} = \left(\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}\right)\left(\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\right).$$

Using the infinite series for e^x, $x=-1$, we get

$$e^{-1} = 1-\frac{1}{1!}+\frac{1}{2!}-\cdots+\frac{(-1)^n}{n!}+\cdots = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}.$$

Thus

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^{m+n}}{m!n!} = \frac{1}{e^2}.$$

Example 6.12. *Prove that*

$$S_n = \sum_{i_1+i_1+\cdots+i_k=n} i_1 i_2 \cdots i_k = \frac{n(n^2-1^2)\cdots(n^2-(k-1)^2)}{(2k-1)!},$$

where the sum is over all possible k-tuples $(i_1, i_2, \ldots, i_k)$ of nonnegative integers that sum to n.

Solution. A moment of thinking shows that we have the equality of formal series

$$\sum_{n=0}^{\infty} S_n x^n = \left(\sum_{m=0}^{\infty} m x^m\right)^k.$$

Now, by differentiating $\sum_{m=0}^{\infty} x^m = \frac{1}{1-x}$ and then multiplying the result by x we get

$$\sum_{m=0}^{\infty} m x^m = \frac{x}{(1-x)^2},$$

hence

$$\sum_{n=0}^{\infty} S_n x^n = \left(\sum_{m=0}^{\infty} m x^m\right)^k = \frac{x^k}{(1-x)^{2k}} = x^k(1-x)^{-2k}.$$

But S_n is precisely the coefficient of x^n in this development, which we can obtain with Newton's binomial formula for the exponent $-2k$. Clearly, $S_n = 0$ for $n \le k-1$ (because, after using the formula we multiply by x^k) – and the result is the expected one. For $n \ge k$ this coefficient is

$$\begin{aligned}
(-1)^{n-k}\binom{-2k}{n-k} &= \frac{(n+k-1)(n+k-2)\cdots(2k+1)(2k)}{(n-k)!} \\
&= \frac{(n+k-1)!}{(2k-1)!(n-k)!} \\
&= \frac{(n+k-1)(n+k-2)\cdots(n-k+2)(n-k+1)}{(2k-1)!} \\
&= \frac{n(n^2-1^2)\cdots(n^2-(k-1)^2)}{(2k-1)!}
\end{aligned}$$

and the formula for S_n is completely proved.

Example 6.13. *Prove that*

$$\sum_{j_1+2j_2+\cdots+nj_n=n} (-1)^{j_1+j_2+\cdots+j_n-1} \frac{(j_1+j_2+\cdots+j_n-1)!}{j_1!j_2!\cdots j_n!} = \frac{1}{n},$$

where the sum is over all the n-tuples $(j_1, j_2, \dots, j_n)$ *of nonnegative integers for which* $j_1 + 2j_2 + \cdots + nj_n = n$.

Solution. The two main ingredients of the proof are the multinomial formula,

$$(x_1 + x_2 + \cdots + x_n)^m = \sum_{j_1+j_2+\cdots+j_n=m} \frac{(j_1+j_2+\cdots+j_n)!}{j_1!j_2!\cdots j_n!} x_1^{j_1} x_2^{j_2} \cdots x_n^{j_n}$$

(which for $n = 2$ gives the binomial theorem), and the logarithmic expansion

$$\ln \frac{1}{1-x} = -\ln(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots = \sum_{k=1}^{\infty} \frac{x^k}{k}.$$

But we can also calculate the logarithm like this:

$$\ln \frac{1}{1-x} = \ln(1 + x + x^2 + \cdots)$$

$$= (x + x^2 + \cdots) - \frac{(x + x^2 + \cdots)^2}{2} + \frac{(x + x^2 + \cdots)^3}{3} - \cdots$$

Because we are interested in the coefficient of x^n (which, in the first form of the development is $1/n$) we can give up the powers of x with exponent greater than n. Thus, the coefficient of x^n is the same as in

$$(x + x^2 + \cdots + x^n) - \frac{(x + x^2 + \cdots + x^n)^2}{2}$$

$$+ \frac{(x + x^2 + \cdots + x^n)^3}{3} - \cdots + (-1)^{n-1} \frac{(x + x^2 + \cdots + x^n)^n}{n}.$$

By the multinomial theorem we have

$$(x + x^2 + \cdots + x^n)^p = \sum_{j_1+j_2+\cdots+j_n=p} \frac{(j_1 + j_2 + \cdots + j_n)!}{j_1!j_2!\cdots j_n!} x^{j_1+2j_2+\cdots+nj_n}$$

hence the coefficient of x^n in this formal series will be

$$\sum_{j_1+2j_2+\cdots+nj_n=n} (-1)^{j_1+j_2+\cdots+j_n-1} \frac{(j_1 + j_2 + \cdots + j_n)!}{j_1!j_2!\cdots j_n!} \frac{1}{j_1 + j_2 + \cdots + j_n}$$

$$= \sum_{j_1+2j_2+\cdots+nj_n=n} (-1)^{j_1+j_2+\cdots+j_n-1} \frac{(j_1 + j_2 + \cdots + j_n - 1)!}{j_1!j_2!\cdots j_n!}.$$

Since first we found this coefficient to be $1/n$, the desired equality follows and our proof is done.
We close this chapter with the classical (and very beautiful) theorem of Lucas which evaluates the binomial coefficients modulo a prime.

Example 6.14. *Let p be a prime, let n and k be positive integers ($n \geq k$), and let $n = n_0 + n_1 p + \cdots + n_s p^s$ and $k = k_0 + k_1 p + \cdots + k_s p^s$ be the base p representations of n and k. Here n_i and k_i are base p digits, that is, they are all from the set $\{0, 1, \ldots, p-1\}$. We then have*

$$\binom{n}{k} \equiv \binom{n_0}{k_0}\binom{n_1}{k_1}\cdots\binom{n_s}{k_s} \pmod{p}.$$

Solution. In the proof we will work in the ring $\mathbb{Z}_p$ of integers modulo p (and, more precisely, we will work in the ring $\mathbb{Z}_p[X]$ of *polynomials* with coefficients in $\mathbb{Z}_p$). The elements of $\mathbb{Z}_p$ are equivalence classes of integers, but we will not use specific notations for the equivalence classes. Instead equivalence classes will just be denoted by numbers. Since we are working in $\mathbb{Z}_p$, we will have equalities rather than with congruences. Thus, for example, the congruence

$$(x + y)^p \equiv x^p + y^p \pmod{p}$$

(proved in the chapter *Mathematical Induction*) will be written

$$(x + y)^p = x^p + y^p,$$

because now x and y actually denote the classes of equivalence of x and y modulo p. In fact this is the starting point of the proof. It is clear that

$$(f+g)^p = f^p + g^p$$

holds for $f, g \in \mathbb{Z}_p[X]$, too (the proof is the same); moreover, we have

$$(f+g)^{p^j} = f^{p^j} + g^{p^j}$$

for any positive integer j (we leave the proof by an easy induction to the careful reader). Thus, in $\mathbb{Z}_p[X]$ we have

$$\begin{aligned}(1+X)^n &= (1+X)^{n_0}((1+X)^p)^{n_1}\cdots((1+X)^{p^s})^{n_s}\\ &= (1+X)^{n_0}(1+X^p)^{n_1}\cdots(1+X^{p^s})^{n_s}\\ &= \left(\sum_{j_0=0}^{n_0}\binom{n_0}{j_0}X^{j_0}\right)\left(\sum_{j_1=0}^{n_1}\binom{n_1}{j_1}X^{j_1p}\right)\cdots\left(\sum_{j_s=0}^{n_s}\binom{n_s}{j_s}X^{j_sp^s}\right).\end{aligned}$$

Now the coefficient of X^k in $(1+X)^n$ is, of course, $\binom{n}{k}$. The expansion of the final product from above contains all terms of the form

$$\binom{n_0}{j_0}\binom{n_1}{j_1}\cdots\binom{n_s}{j_s}X^{j_0+j_1p+\cdots+j_sp^s},$$

with $0 \le j_i \le n_i$ for $0 \le i \le s$. Since the base p representation of a natural number is unique, X^k only appears in this expansion when $j_i = k_i$ for every $0 \le i \le s$, therefore it has the coefficient

$$\binom{n_0}{k_0}\binom{n_1}{k_1}\cdots\binom{n_s}{k_s}.$$

By equating the two coefficients of X^k, we obtain that (in $\mathbb{Z}_p$)

$$\binom{n}{k} = \binom{n_0}{k_0}\binom{n_1}{k_1}\cdots\binom{n_s}{k_s},$$

which, if we consider numbers instead of residue classes, is precisely the congruence stated by Lucas's theorem.

Note that, as an immediate consequence of Lucas's theorem we have the fact that $\binom{n}{k}$ is divisible by p if and only if there exists $0 \le i \le s$ such that $n_i < k_i$.

Chapter 7

Sums and Products in Number Theory

There are a lot of problems related to sums and products in Number Theory. An area which we would like you to cover is based on three well-known functions that on positive integers n have relatively short formulas in terms of their prime factorization

$$n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_n^{\alpha_n}.$$

These are

• $\tau(n)$ – the number of positive divisors of n.

We can write this as $\tau(n) = \sum\limits_{d|n} 1$ or in terms of the prime factorization we have

$$\tau(n) = (\alpha_1 + 1)(\alpha_2 + 1) \cdots (\alpha_n + 1).$$

• $\sigma(n)$ – the sum of the positive divisors of n.

We can rewrite this as $\sigma(n) = \sum\limits_{d|n} d$ or in terms of the prime factorization we have

$$\sigma(n) = \frac{p_1^{\alpha_1+1} - 1}{p_1 - 1} \cdot \frac{p_2^{\alpha_2+1} - 1}{p_2 - 1} \cdots \frac{p_n^{\alpha_n+1} - 1}{p_n - 1}.$$

• $\phi(n)$ – the Euler's totient function, the number of all positive integers less than or equal to n that are relatively prime to n. (The "equal" case obviously only matters for $n = 1$.) In terms of the prime factorization of n we have

$$\phi(n) = n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)\cdots\left(1 - \frac{1}{p_n}\right)$$

and we also have $\sum\limits_{d|n} \phi(d) = n$.

The last fact is not so obvious and we would like to prove it. We consider the rational numbers

$$\frac{1}{n}, \frac{2}{n}, \ldots, \frac{n}{n}.$$

Clearly, there are n numbers in the list. On the other hand we can obtain a new list by reducing each number in the above list to the lowest terms; that is express each fraction as a quotient of relatively prime integers. The denominators of the numbers in the new list will all be the divisors of n. If $d \mid n$, exactly $\phi(d)$ of the numbers in the list will have d as their denominator (this is the meaning of lowest terms!). Hence, there are $\sum\limits_{d|n} \phi(d)$ terms in the new list. Because the two lists have the same number of terms, we obtain the desired result. We continue with a few easy problems.

Example 7.1. *Let $\tau(n)$ be the number of positive divisors of n. Prove that*

$$\prod_{d|n} d = n^{\frac{\tau(n)}{2}}.$$

Solution. Due to the fact that, when d runs through all (positive) divisors of n, n/d also runs through the set of all (positive) divisors of n, we have

$$\prod_{d|n} d = \prod_{d|n} \frac{n}{d} = \frac{n^{\tau(n)}}{\prod\limits_{d|n} d},$$

therefore

$$\left(\prod_{d|n} d\right)^2 = n^{\tau(n)},$$

and the conclusion follows.
Note that, if $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$ is the prime factorization of n, then any divisor of n has the form $p_1^{\beta_1} \cdots p_k^{\beta_k}$, with $\beta_i \in \{0, 1, \ldots, \alpha_i\}$ for every $1 \leq i \leq k$. This immediately yields the formula (that we already presented)

$$\tau(n) = (\alpha_1 + 1) \cdots (\alpha_k + 1),$$

and shows that the equality stated by the problem can also be read as

$$\prod_{0 \leq \beta_1 \leq \alpha_1, \ldots, 0 \leq \beta_k \leq \alpha_k} p_1^{\beta_1} \cdots p_k^{\beta_k} = \left(p_1^{\alpha_1} \cdots p_k^{\alpha_k}\right)^{\frac{(\alpha_1+1)\cdots(\alpha_k+1)}{2}}$$

(the product from the left-hand side is over all possible choices of $(\beta_1, \ldots, \beta_k) \in \{0, 1, \ldots, \alpha_1\} \times \cdots \times \{0, 1, \ldots, \alpha_k\}$). Try to prove it in this form, too! (You have to prove that the exponents of each p_i in both sides are equal; the problem becomes a – not very complicated – counting problem if we put it this way.)

Example 7.2. *For positive integers m and n prove that*

$$\prod_{d|m} d = \prod_{d|n} d$$

implies $m = n$.

Solution. By the previous example, we are given that

$$m^{\tau(m)} = n^{\tau(n)},$$

which immediately yields that m and n have precisely the same prime factors in their factorizations. So, let $p_1, \ldots, p_k$ be these factors, and let

$$m = p_1^{a_1} \cdots p_k^{a_k} \quad \text{and} \quad n = p_1^{b_1} \cdots p_k^{b_k}.$$

By comparing the exponents in the above equality we get

$$\frac{a_1}{b_1} = \cdots = \frac{a_k}{b_k} = \frac{\tau(n)}{\tau(m)} = \frac{(b_1+1)\cdots(b_k+1)}{(a_1+1)\cdots(a_k+1)}.$$

From these equalities we see that we cannot have $a_1 > b_1$ because the equality of ratios would imply $a_i > b_i$ for all $1 \le i \le k$. But then

$$(a_1+1)\cdots(a_k+1) > (b_1+1)\cdots(b_k+1)$$

and we would get

$$1 < \frac{a_1}{b_1} = \cdots = \frac{a_k}{b_k} = \frac{\tau(n)}{\tau(m)} = \frac{(b_1+1)\cdots(b_k+1)}{(a_1+1)\cdots(a_k+1)} < 1,$$

which is clearly impossible. Similarly, $a_1 < b_1$ leads to contradiction, so it follows that $a_1 = b_1$, which implies $a_i = b_i$ for all i, and, consequently, $m = n$.

Example 7.3. *Find the sum of all positive integers less than n and relatively prime to n.*

Solution. If $n \ge 2$, then the sum is $n\phi(n)/2$, with $\phi(n)$ representing the number of the positive integers less than n and relatively prime to n. The key observation is that if $k < n$ and k is relatively prime to n, then $n - k$ is also less than n and relatively prime to n. Consequently,

$$\sum_{1\le k<n,(k,n)=1} k = \sum_{1\le k<n,(k,n)=1} (n-k)$$

$$= n \sum_{1\le k<n,(k,n)=1} 1 - \sum_{1\le k<n,(k,n)=1} k = n\phi(n) - \sum_{1\le k<n,(k,n)=1} k$$

$$\Rightarrow \sum_{1\le k<n,(k,n)=1} k = \frac{n\phi(n)}{2},$$

as we claimed.

Example 7.4. *Let $n = p_1^{\alpha_1}\cdots p_k^{\alpha_k}$ be the prime factorization of the positive integer $n \ge 2$, and let $0 < a_1 < \cdots < a_{\phi(n)} < n$ be all the positive integers relatively prime to n that are less than n. Prove that*

$$a_1^2 + a_2^2 + \cdots + a_{\phi(n)}^2 = \frac{1}{6}\phi(n)(2n^2 + (-1)^k p_1\cdots p_k).$$

Solution. Let $A=\{1,2,\ldots,n\}$ and let $A_i=\{a\in A \mid p_i \text{ divides } a\}$, $1\le i\le k$. For a subset M of A we denote

$$S(M)=\sum_{x\in M} x^2.$$

Then

$$S(A)=\sum_{i=1}^{n} i^2=\frac{n(n+1)(2n+1)}{6}=\frac{1}{3}n^3+\frac{1}{2}n^2+\frac{1}{6}n$$

and

$$S(A_i)=p_i^2+(2p_i)^2+\cdots+\left(\frac{n}{p_i}p_i\right)^2=p_i^2\left[\frac{1}{3}\left(\frac{n}{p_i}\right)^3+\frac{1}{2}\left(\frac{n}{p_i}\right)^2+\frac{1}{6}\left(\frac{n}{p_i}\right)\right].$$

Analogously,

$$S(A_i\cap A_j)=p_i^2p_j^2\left[\frac{1}{3}\left(\frac{n}{p_ip_j}\right)^3+\frac{1}{2}\left(\frac{n}{p_ip_j}\right)^2+\frac{1}{6}\left(\frac{n}{p_ip_j}\right)\right],\ \forall\ i\neq j$$

and so on for $S(A_i\cap A_j\cap A_k)$, etc. Using the inclusion-exclusion principle we get

$$a_1^2+a_2^2+\cdots+a_{\phi(n)}^2=S(A)-S\left(\bigcup_{i=1}^{k} A_i\right)$$

$$=\frac{1}{3}n^3+\frac{1}{2}n^2+\frac{1}{6}n-\sum_{i=1}^{k} p_i^2\left[\frac{1}{3}\left(\frac{n}{p_i}\right)^3+\frac{1}{2}\left(\frac{n}{p_i}\right)^2+\frac{1}{6}\left(\frac{n}{p_i}\right)\right]$$

$$+\sum_{1\le i<j\le k}(p_ip_j)^2\left[\frac{1}{3}\left(\frac{n}{p_ip_j}\right)^3+\frac{1}{2}\left(\frac{n}{p_ip_j}\right)^2+\frac{}{6}\left(\frac{n}{p_ip_j}\right)\right]-\cdots$$

$$+(-1)^k(p_1\cdots p_k)^2\left[\frac{1}{3}\left(\frac{n}{p_1\cdots p_k}\right)^3+\frac{1}{2}\left(\frac{n}{p_1\cdots p_k}\right)^2+\frac{1}{6}\left(\frac{n}{p_1\cdots p_k}\right)\right].$$

Thus we can write $a_1^2 + a_2^2 + \cdots + a_{\phi(n)}^2$ as $\alpha n^3 + \beta n^2 + \gamma n$, where

$$\alpha = \frac{1}{3}\left(1 - \sum_{i=1}^{k} \frac{1}{p_i} + \sum_{i<j} \frac{1}{p_i p_j} - \cdots + (-1)^k \frac{1}{p_1 \cdots p_k}\right),$$

$$\begin{aligned}\beta &= \frac{1}{2}\left(1 - \sum_{i=1}^{k} 1 + \sum_{i<j} 1 - \cdots + (-1)^k 1\right) \\ &= \frac{1}{2}\left(\binom{k}{0} - \binom{k}{1} + \binom{k}{2} - \cdots + (-1)^k \binom{k}{k}\right),\end{aligned}$$

$$\gamma = \frac{1}{6}\left(1 - \sum_{i=1}^{k} p_i + \sum_{i<j} p_i p_j - \cdots + (-1)^k p_1 \cdots p_k\right).$$

This means that

$$\begin{aligned}\alpha &= \frac{1}{3}\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)\cdots\left(1 - \frac{1}{p_n}\right) = \frac{1}{3} \cdot \frac{\phi(n)}{n}, \\ \beta &= \frac{1}{2}(1-1)^k = 0, \\ \gamma &= \frac{1}{6}(1-p_1)(1-p_2)\cdots(1-p_k) = \frac{(-1)^k}{6} p_1 \cdots p_k \cdot \frac{\phi(n)}{n},\end{aligned}$$

and combining all these gives us the desired result.

Example 7.5. *Prove Euler's theorem, namely that $a^{\phi(n)} - 1$ is divisible by n for every integer n such that a and n are relatively prime.*

Solution. Let $1 \le x_1 < \cdots < x_{\phi(n)} < n$ be the $\phi(n)$ integers which are less than n and relatively prime to n. Since a is relatively prime to n, the numbers $ax_1, \ldots, ax_{\phi(n)}$ are also relatively prime to n. Further they represent distinct congruence classes modulo n since if $ax_i \equiv ax_j \pmod{n}$, then $n | a(x_j - x_i)$ and since a is relatively prime to n, this forces $n | x_j - x_i$ and hence $x_i = x_j$. Thus the numbers $ax_1, \ldots, ax_{\phi(n)}$ are modulo n some reordering of the numbers $x_1, \ldots, x_{\phi(n)}$ and in particular

$$(ax_1)\cdots(ax_{\phi(n)}) = a^{\phi(n)} x_1 \cdots x_{\phi(n)} \equiv x_1 \cdots x_{\phi(n)} \pmod{n}.$$

Since the x_i are all relatively prime to n, we may cancel them off to get $a^{\phi(n)} \equiv 1 \pmod{n}$ or $a^{\phi(n)} - 1$ is divisible by n.
In particular, for a prime p and an integer a which is not divisible by p (that is, a and p are relatively prime), we get that $a^{\phi(p)} - 1 = a^{p-1} - 1$ is divisible by p. Consequently, $a^p - a = a(a^{p-1} - 1)$ is divisible by p for every integer a – and we arrived again to Fermat's little theorem (see also the chapter *Mathematical Induction*).
Let us introduce a new function, called Möbius function. This function plays an important role in proving some deep results in Number Theory. We begin with the notions of arithmetic function and multiplicative function.
Arithmetic functions are functions defined on the positive integers that are complex valued. The arithmetic function $f \neq 0$ is called **multiplicative** if for any relatively prime positive integers m and n,

$$f(mn) = f(m)f(n).$$

Note that if f is multiplicative, then $f(1) = 1$. Indeed if a is a positive integer for which $f(a) \neq 0$, then $f(a) = f(a \cdot 1) = f(a)f(1)$, and simplifying by $f(a)$ yields $f(1) = 1$. Note also that if f is multiplicative and $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$ is the prime factorization of the positive integer n, then

$$f(n) = f(p_1^{\alpha_1}) \cdots f(p_k^{\alpha_k}).$$

One reason why we discuss multiplicative functions is that $\tau(n)$, $\sigma(n)$, $\phi(n)$ are multiplicative functions.
The fourth arithmetic function we present is the Möbius function defined by

$$\mu(n) = \begin{cases} 1 & \text{if} \quad n = 1, \\ 0 & \text{if} \quad p^2 \mid n \text{ for some prime } p > 1, \\ (-1)^k & \text{if} \quad n = p_1 \cdots p_k, \text{ where } p_1, p_2, \ldots, p_k \text{ are distinct primes.} \end{cases}$$

For example, $\mu(2) = -1$, $\mu(6) = 1$, $\mu(12) = \mu(2^2 \cdot 3) = 0$.

Theorem 1. *The Möbius function is multiplicative.*
Proof. Let m, n be positive integers such that $\gcd(m, n) = 1$. If $p^2 \mid m$ for some $p > 1$, then $p^2 \mid mn$ and so $\mu(m) = \mu(mn) = 0$ and we are done. Similarly if $p^2 \mid n$ we are done.

Otherwise, $m = p_1 \cdots p_k$, $n = q_1 \cdots q_h$, where $p_1, \ldots p_k, q_1, \ldots q_h$ are distinct primes. Then $\mu(m) = (-1)^k$, $\mu(n) = (-1)^h$, and $mn = p_1 \cdots p_k q_1 \cdots q_h$.
It follows that

$$\mu(mn) = (-1)^{k+h} = (-1)^k (-1)^h = \mu(m)\mu(n).$$

Another important property of the Möbius function is that the sum of its values over all divisors of a given positive integer can be easily evaluated.

Example 7.6. *We have*

$$\sum_{d|n} \mu(d) = \begin{cases} 1 & \textit{if } n = 1 \\ 0, & \textit{if } n > 1. \end{cases}$$

Solution. This is clear for $n = 1$, when there is only one term in the sum (namely $\mu(1)$). For $n > 1$, if $p_1, \ldots, p_k$ are all the distinct prime factors of n, we realize after a moment of thinking, that

$$\begin{aligned}\sum_{d|n} \mu(d) &= \mu(1) + \sum_{1 \le i \le k} \mu(p_i) + \sum_{1 \le i < j \le k} \mu(p_i p_j) + \cdots + \mu(p_1 p_2 \cdots p_k) \\ &= \binom{k}{0} - \binom{k}{1} + \binom{k}{2} - \cdots + (-1)^k \binom{k}{k} = 0\end{aligned}$$

by the definition of μ (the divisors of n that are divisible by some square do not contribute to the sum).
Or, we can consider the summation function defined by

$$F(n) = \sum_{d|n} \mu(d)$$

and prove that it is multiplicative (see Theorem 2 below). Because

$$F(p^\alpha) = \mu(1) + \mu(p) + \cdots + \mu(p^\alpha) = 1 - 1 = 0$$

for every prime number p and any positive integer α, we immediately get $F(n) = 0$ for $n \ge 2$, based on the multiplicative property of F. Of course, $F(1) = 1$ follows immediately.
By using this result we can immediately solve the following exercise.

Example 7.7. *For any real number $x \geq 1$ we have*

$$\sum_{k=1}^{\infty} \mu(k) \left\lfloor \frac{x}{k} \right\rfloor = 1.$$

Solution. This is not actually an infinite sum, as all floors are 0 for $k > x$. We have

$$\begin{aligned}
\sum_{k=1}^{\infty} \mu(k) \left\lfloor \frac{x}{k} \right\rfloor &= \sum_{1 \leq k \leq x} \mu(k) \sum_{1 \leq j \leq x/k} 1 \\
&= \sum_{k=1}^{\infty} \mu(k) \sum_{jk \leq x} 1 \\
&= \sum_{1 \leq n \leq x} \sum_{k|n} \mu(k) = 1
\end{aligned}$$

We changed the order of summation by putting together, for every $1 \leq n \leq x$, all terms for which $jk = n$, then we used (as we showed in the previous example)

$$\sum_{k|n} \mu(k) = \begin{cases} 1, & n = 1 \\ 0, & n > 1. \end{cases}$$

Not only for the Möbius function, but for any arithmetic function f we can consider its **summation function** F defined by

$$F(n) = \sum_{d|n} f(d).$$

One connection between f and F is given by the following result.

Theorem 2. *If f is multiplicative, then so is its summation function F.*

Proof. Let m and n be positive relatively prime integers and let d be a divisor of mn. Then d can be uniquely represented as $d = kh$, where $k \mid m$ and $h \mid n$.

Because $\gcd(m,n)=1$, we have $\gcd(k,h)=1$, so $f(kh)=f(k)f(h)$. Hence

$$\begin{aligned} F(mn) &= \sum_{d|mn} f(d) = \sum_{k|m,h|n} f(k)f(h) \\ &= \left(\sum_{k|m} f(k)\right)\left(\sum_{h|n} f(h)\right) \\ &= F(m)F(n), \end{aligned}$$

and we are done.
Note that if f is a multiplicative function and $n=p_1^{\alpha_1}\cdots p_k^{\alpha_k}$, then

$$F(n)=F(p_1^{\alpha_1})\cdot F(p_2^{\alpha_2})\cdots F(p_n^{\alpha_n})=\prod_{i=1}^{k}\left(1+f(p_i)+f(p_i^2)+\cdots+f(p_i^{\alpha_i})\right).$$

The last formula is pretty useful (although almost evident) and helps to solve many problems related to the Möbius function, for instance:

Example 7.8. *If f is a multiplicative function and $n=p_1^{\alpha_1}\cdots p_k^{\alpha_k}$, then*

$$\sum_{d|n}\mu(d)f(d)=(1-f(p_1))(1-f(p_2))\cdots(1-f(p_k)).$$

Solution. Consider the function

$$G(n)=\sum_{d|n}\mu(d)f(d),$$

which is multiplicative. Using the above result we get

$$G(n)=G(p_1^{\alpha_1})\cdots G(p_k^{\alpha_k}),$$

where

$$G(p_i^{\alpha_i})=\sum_{d|p_i^{\alpha_i}}\mu(d)f(d)=\mu(1)f(1)-\mu(p_i)f(p_i)=1-f(p_i),$$

and the conclusion follows.

Example 7.9. *Let* $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$. *Prove that*

$$\sum_{d|n} \mu(d)\sigma(d) = (-1)^k p_1 p_2 \cdots p_k.$$

Solution. Using the previous example, we get

$$\sum_{d|n} \mu(d) f(d) = (1 - f(p_1))(1 - f(p_2)) \cdots (1 - f(p_k)).$$

Let us take $f(n) = \sigma(n) = \sum_{d|n} d$. Then this formula becomes

$$\begin{aligned}\sum_{d|n} \mu(d) f(d) &= (1 - (p_1 + 1))(1 - (p_2 + 1)) \cdots (1 - (p_n + 1)) \\ &= (-1)^k p_1 p_2 \cdots p_k.\end{aligned}$$

We also have another beautiful result related to the Möbius function it is the Möbius inversion formula.

Theorem 3. (Möbius inversion formula) *Let* f *be a multiplicative function. Then*

$$f(n) = \sum_{d|n} \mu(d) F\left(\frac{n}{d}\right).$$

Proof. We have

$$\begin{aligned}\sum_{d|n} \mu(d) F\left(\frac{n}{d}\right) &= \sum_{d|n} \mu(d) \left(\sum_{c|\frac{n}{d}} f(c)\right) = \sum_{d|n} \left(\sum_{c|\frac{n}{d}} \mu(d) f(c)\right) \\ &= \sum_{c|n} \left(\sum_{d|\frac{n}{c}} \mu(d) f(c)\right) = \sum_{c|n} f(c) \left(\sum_{d|\frac{n}{c}} \mu(d)\right).\end{aligned}$$

Since for $n/c > 1$ we have $\sum_{d|\frac{n}{c}} \mu(d) = 0$. The only nonzero term in this last sum is the $c = n$ term which is equal to $f(n)$.

To complete our journey we present the last theorem that connects a function with its summation function:

Theorem 4. *Let f be an arithmetic function and let F be its summation function. If F is multiplicative, then so is f.*

Proof. Let m, n be positive integers such that $\gcd(m,n) = 1$ and let d be a divisor of mn. Then $d = kh$, where $k \mid m, h \mid n$, and $\gcd(k,h) = 1$. Applying the Möbius formula it follows

$$\begin{aligned} f(mn) &= \sum_{d \mid mn} \mu(d) F\left(\frac{mn}{d}\right) = \sum_{k \mid m, h \mid n} \mu(kh) F\left(\frac{mn}{kh}\right) \\ &= \sum_{k \mid m, h \mid n} \mu(k)\mu(h) F\left(\frac{m}{k}\right) F\left(\frac{n}{h}\right) \\ &= \left(\sum_{k \mid m} \mu(k) F\left(\frac{m}{k}\right)\right)\left(\sum_{h \mid n} \mu(h) F\left(\frac{n}{h}\right)\right) = f(m)f(n). \end{aligned}$$

We also encourage the reader to redevelop some properties of the functions $\tau(n)$, $\sigma(n)$, $\phi(n)$ by using the general results we proved above. The following problem is an example of how we can deal with such type of problems.

Example 7.10. *For any positive integer n prove that*

$$\sum_{d \mid n} \tau(d) \mu\left(\frac{n}{d}\right) = 1.$$

Solution. Consider the function $F(n)$ which is the summation function for $f(n) = 1$, we have

$$F(n) = \sum_{d \mid n} f(d) = \sum_{d \mid n} 1 = \tau(n).$$

Writing the Möbius inversion formula for $f(n)$ we get

$$1 = f(n) = \sum_{d \mid n} \tau(d) \mu\left(\frac{n}{d}\right).$$

As you can see the problems are not so hard, but you need to get used to them and the best way to do that is to practice, to solve a number of problems on this topic.

Chapter 8

Problems

1 Easy Problems

E1. Evaluate
$$\frac{(6!+5!)(5!+4!)(4!+3!)(3!+2!)(2!+1!)}{(6!-5!)(5!-4!)(4!-3!)(3!-2!)(2!-1!)}.$$

E2. Prove that
$$1\cdot n+2\cdot(n-1)+3\cdot(n-2)+\cdots+(n-1)\cdot 2+n\cdot 1=\frac{n(n+1)(n+2)}{6}.$$

E3. Find a general formula for the sum
$$1+11+111+\cdots+\underbrace{11\ldots11}_{n \text{ digits}}.$$

E4. Evaluate the product
$$\prod_{k=2}^{n}\frac{4k^3-3k+1}{4k^3-3k-1}.$$

E5. For a fixed positive integer n let $a_k=2^{2^{k-n}}+k$, $k=0,1,\ldots,n$. Prove that
$$(a_1-a_0)\cdots(a_n-a_{n-1})=\frac{7}{a_1+a_0}.$$

E6. Prove that for all positive integers n

$$\sum_{k=1}^{n} \frac{k}{(k+1)!} < 1.$$

E7. Evaluate

$$\sum_{k=1}^{n} \frac{k+1}{(k-1)! + k! + (k+1)!}.$$

E8. Evaluate

$$\prod_{k=1}^{n} \left(\frac{1}{8} + \frac{k+1}{(2k+1)^2} \right).$$

E9. Evaluate

$$\sum_{n \geq 2} \frac{3n^2+1}{(n^3-n)^3}.$$

E10. Evaluate

$$\sum_{k=1}^{n} k!(k^2+1).$$

E11. Consider n arithmetic progressions with the same common difference d and having their first terms $1, 2, 3, \ldots, n$. If $S(n,k)$ is the sum of the first n terms of the arithmetic progression that has its first term k, prove that

$$S(n,1) + S(n,2) + S(n,3) + \cdots + S(n,n) = \frac{n^2}{2}(2 + (n-1)(d+1)).$$

E12. Let $a_1, a_2, \ldots, a_n$ be an arithmetic progression with $a_k \neq 0$ for all $1 \leq k \leq n$. Prove that

$$\sum_{k=1}^{n-1} \frac{1}{a_k a_{k+1}} = \frac{n-1}{a_1 a_n}.$$

E13. Let $a_1, a_2, \ldots, a_n$ be an arithmetic progression with positive terms. Prove that

$$\frac{1}{\sqrt{a_1}+\sqrt{a_2}}+\frac{1}{\sqrt{a_2}+\sqrt{a_3}}+\cdots+\frac{1}{\sqrt{a_{n-1}}+\sqrt{a_n}}=\frac{n-1}{\sqrt{a_1}+\sqrt{a_n}}.$$

E14. Prove the inequality

$$\frac{1}{\sqrt{1}+\sqrt{3}}+\frac{1}{\sqrt{5}+\sqrt{7}}+\cdots+\frac{1}{\sqrt{9997}+\sqrt{9999}}>24.$$

E15. Evaluate

$$\sum_{k=1}^{n}\frac{k^2-\frac{1}{2}}{k^4+\frac{1}{4}}.$$

E16. Prove that

$$\sum_{k=1}^{9999}\frac{1}{\left(\sqrt{k}+\sqrt{k+1}\right)\left(\sqrt[4]{k}+\sqrt[4]{k+1}\right)}=9.$$

E17. Find the closed form

$$\sqrt{1+\frac{1}{1^2}+\frac{1}{2^2}}+\sqrt{1+\frac{1}{2^2}+\frac{1}{3^2}}+\cdots+\sqrt{1+\frac{1}{1999^2}+\frac{1}{2000^2}}.$$

E18. Evaluate the sum

$$\sum_{k=1}^{n}\frac{1}{\sqrt{2k+\sqrt{4k^2-1}}}.$$

E19. Let $a_n=\sqrt{1+\left(1+\frac{1}{n}\right)^2}+\sqrt{1+\left(1-\frac{1}{n}\right)^2}$, $n\geq 1$. Prove that

$$\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_{20}}$$

is a positive integer.

E20. Prove that

$$\sum_{k=0}^{n} \frac{1}{k!(n-k)!} = \frac{2^n}{n!}.$$

E21. Evaluate

$$\frac{\binom{n}{0}}{1} + \frac{\binom{n}{1}}{2} + \cdots + \frac{\binom{n}{n}}{n+1}.$$

E22. Prove that

$$\binom{n}{m} + \binom{n-1}{m-1} + \binom{n-2}{m-2} + \cdots + \binom{n-m}{0} = \binom{n+1}{m}.$$

E23. Evaluate

$$\binom{2008}{3} - 2\binom{2008}{4} + 3\binom{2008}{5} - 4\binom{2008}{6} + \cdots - 2004\binom{2008}{2006} + 2005\binom{2008}{2007}.$$

E24. Evaluate the sum

$$\sum_{k=1}^{\infty} \frac{6^k}{(3^k - 2^k)(3^{k+1} - 2^{k+1})}.$$

E25. Let F_n be the n^{th} Fibonacci number ($F_1 = F_2 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for all $n \geq 3$). Evaluate

$$\sum_{k=2}^{\infty} \frac{F_k}{F_{k-1}F_{k+1}}.$$

E26. Prove that for all $n \geq 3$,

$$\prod_{k=2}^{n-1} \left(\frac{1}{9} + \frac{k^2+k+1}{(k-1)^3} \right) = \frac{1}{3^{2n-1}} \left(\frac{n^3-n}{2} \right)^3.$$

E27. Let i denote the imaginary unit. Evaluate

$$\prod_{k=1}^{n} \frac{1+i+k(k+1)}{1-i+k(k+1)}.$$

E28. Prove that the identity

$$(x+y+z)^5-(x^5+y^5+z^5) = 5(x+y)(x+z)(y+z)(x^2+y^2+z^2+xy+xz+yz)$$

holds for any numbers x, y, and z.

E29. Evaluate the following sum for every positive integer n

$$1+\cos\frac{\pi}{n}+\cos\frac{2\pi}{n}+\cdots+\cos\frac{(n-1)\pi}{n}.$$

E30. Evaluate

$$\sum_{k=1}^{n}(-1)^{\frac{k(k+1)}{2}}.$$

E31. Evaluate

$$\sum_{k=1}^{n}(-1)^{\frac{k(k+1)}{2}}k.$$

E32. Prove that

$$\sum_{k=1}^{n} \frac{(-1)^{k+1}}{1^2-2^2+3^2-\cdots+(-1)^{k+1}k^2} = \frac{2n}{n+1}.$$

E33. If $r_1, r_2, \ldots, r_n$ and $t_1, t_2, \ldots, t_n$ are real numbers, prove that

$$\sum_{k=1}^{n}\sum_{l=1}^{n} r_k r_l \cos(t_k - t_l) \geq 0.$$

E34. Prove that

$$\left(\sqrt{3}+\tan 1^\circ\right)\left(\sqrt{3}+\tan 2^\circ\right)\cdots\left(\sqrt{3}+\tan 29^\circ\right) = 2^{29}.$$

E35. Evaluate
$$(1-\cot 1^\circ)(1-\cot 2^\circ)\cdots(1-\cot 44^\circ).$$

E36. Prove that
$$\left(1-\frac{\cos 61^\circ}{\cos 1^\circ}\right)\left(1-\frac{\cos 62^\circ}{\cos 2^\circ}\right)\cdots\left(1-\frac{\cos 119^\circ}{\cos 59^\circ}\right)=1.$$

E37. Prove that for every integer $n>1$,
$$\cos\frac{2\pi}{2^n-1}\cos\frac{4\pi}{2^n-1}\cdots\cos\frac{2^n\pi}{2^n-1}=\frac{1}{2^n}.$$

E38. Let n be a given positive integer and let
$$a_k=2\cos\frac{\pi}{2^{n-k}},\ k=0,1,\ldots,n-1.$$
Prove that
$$\prod_{k=0}^{n-1}(1-a_k)=\frac{(-1)^{n-1}}{1+a_0}.$$

E39. The sequence $\{x_n\}_{n\geq 1}$ is defined by
$$x_1=\frac{1}{2},\ x_{k+1}=x_k^2+x_k.$$
Find the greatest integer less than
$$\frac{1}{x_1+1}+\frac{1}{x_2+1}+\cdots+\frac{1}{x_{100}+1}.$$

E40. Solve the problem left unsolved in the *Introduction*. Namely, if n is any given positive integer and f is defined by $f(x)=x-\left\lfloor\frac{x}{2}\right\rfloor$ show by telescoping that
$$\sum_{k=0}^{\infty}\left\lfloor\frac{f^{[k]}(n)}{2}\right\rfloor=n-1,$$
where $f^{[k]}$ is the kth iterate of f (that is, $f^{[k]}=f\circ f\circ\cdots\circ f$ with k appearances of f; we also consider $f^{[0]}$ to be the identity function that maps x to x, for every x).

2 Medium Problems

M1. For each positive integer k, let $f(k) = 4^k + 6^k + 9^k$. Prove that for all nonnegative integers m and n, $f(2^m)$ divides $f(2^n)$ whenever m is less than or equal to n.

M2. Evaluate

$$1^2 + 2^2 + 3^2 - 4^2 - 5^2 + 6^2 + 7^2 + 8^2 - 9^2 - 10^2 + \cdots - 2010^2,$$

where each three consecutive $+$ signs are followed by two $-$ signs.

M3. Prove that

$$1 + 2q + 3q^2 + \cdots + nq^{n-1} = \frac{1 - nq^n}{1 - q} + \frac{q - q^n}{(1 - q)^2},$$

for every $q \neq 1$.

M4. Evaluate

$$\sum_{k=1}^{n} \frac{k}{k^4 + k^2 + 1}.$$

M5. Evaluate the sum

$$\frac{1}{3+1} + \frac{2}{3^2+1} + \frac{2^2}{3^4+1} + \cdots + \frac{2^n}{3^{2^n}+1}.$$

M6. Let $f_n = 2^{2^n} + 1$, $n = 1, 2, 3, \ldots$ Prove that

$$\frac{1}{f_1} + \frac{2}{f_2} + \cdots + \frac{2^{n-1}}{f_n} < \frac{1}{3},$$

for all positive integers n.

M7. Let $a_n = 3n + \sqrt{n^2 - 1}$ and $b_n = 2(\sqrt{n^2 - n} + \sqrt{n^2 + n})$, $n \geq 1$. Prove that

$$\sqrt{a_1 - b_1} + \sqrt{a_2 - b_2} + \cdots + \sqrt{a_{49} - b_{49}} = A + B\sqrt{2},$$

for some integers A and B.

M8. Let $m \leq n$ be positive integers. Prove the double inequality

$$2\left(\sqrt{n+1}-\sqrt{m}\right) < \frac{1}{\sqrt{m}} + \frac{1}{\sqrt{m+1}} + \cdots + \frac{1}{\sqrt{n-1}} + \frac{1}{\sqrt{n}} < 2\left(\sqrt{n}-\sqrt{m-1}\right).$$

M9. Let

$$a_n = 2 - \frac{1}{n^2 + \sqrt{n^4 + \frac{1}{4}}}, \quad n = 1, 2, \ldots.$$

Prove that $\sqrt{a_1} + \sqrt{a_2} + \cdots + \sqrt{a_{119}}$ is an integer.

M10. Prove that there is no positive integer n for which

$$\prod_{k=1}^{n}(k^4 + k^2 + 1)$$

is a perfect square.

M11. Let F_n be the nth Fibonacci number. Prove that

$$\prod_{k=0}^{n}(F_{2^k-1} + F_{2^k+1}) = F_{2^{n+1}}.$$

M12. Let x be a real number in the interval $(-1, 1)$. Evaluate

$$\prod_{k=0}^{\infty}(1 - x^{2^k} + x^{2^{k+1}}).$$

M13. Let F_n be the n^{th} Fibonacci number. Evaluate

$$\sum_{k=2}^{\infty} \frac{1}{F_{k-1}F_{k+1}}.$$

M14. Let n be a nonnegative integer. Prove that

$$\sum_{k=1}^{n} k\binom{n}{k}^2 = n\binom{2n-1}{n-1}.$$

M15. Let n be an odd integer greater than or equal to 5. Prove that

$$\binom{n}{1} - 5\binom{n}{2} + 5^2\binom{n}{3} - \cdots + (-1)^{n-1}5^{n-1}\binom{n}{n}$$

is not a prime number.

M16. Prove that for any positive integer n the number

$$a_n = \binom{2n+1}{0}2^{2n} + \binom{2n+1}{2}2^{2n-2}\cdot 3 + \cdots + \binom{2n+1}{2n}3^n$$

is the sum of two consecutive perfect squares.

M17. Let n be a positive integer and a be a real number, such that $\frac{a}{\pi}$ is an irrational number. Evaluate

$$\frac{1}{\cos a - \cos 3a} + \frac{1}{\cos a - \cos 5a} + \cdots + \frac{1}{\cos a - \cos(2n+1)a}.$$

M18. Prove that

$$\frac{1}{\sin 45^\circ \sin 46^\circ} + \frac{1}{\sin 47^\circ \sin 48^\circ} + \cdots + \frac{1}{\sin 133^\circ \sin 134^\circ} = \frac{1}{\sin 1^\circ}.$$

M19. Prove that for every positive integer n and for every real number $x \neq \frac{s\pi}{2^t}$ $(t = 0, 1, 2, \ldots, n,\ s$ an integer),

$$\sum_{k=1}^{m} \frac{1}{\sin 2^k x} = \cot x - \cot 2^n x.$$

M20. Show that

$$\frac{\sin x}{\cos x}+\frac{\sin 2x}{\cos^2 x}+\cdots+\frac{\sin nx}{\cos^n x}=\cot x-\frac{\cos(n+1)x}{\sin x\cos^n x},$$

for all $x \neq s\frac{\pi}{2}$, where s is an integer.

M21. For each positive integer number n prove that

$$\cos\frac{2\pi}{2n+1}+\cos\frac{4\pi}{2n+1}+\cdots+\cos\frac{2n\pi}{2n+1}=-\frac{1}{2}.$$

M22. Let $\zeta \neq 1$ be a complex number with $\zeta^{23}=1$. Evaluate

$$\sum_{k=0}^{22}\frac{1}{1+\zeta^k+\zeta^{2k}}.$$

M23. Prove that

$$\lfloor x\rfloor+\left\lfloor x+\frac{1}{n}\right\rfloor+\left\lfloor x+\frac{2}{n}\right\rfloor+\cdots+\left\lfloor x+\frac{n-1}{n}\right\rfloor=\lfloor nx\rfloor,$$

for all $x \in \mathbb{R}$ and any positive integer n.

M24. Prove that for every positive integer n

$$\left\lfloor\frac{n+2^0}{2^1}\right\rfloor+\left\lfloor\frac{n+2^1}{2^2}\right\rfloor+\left\lfloor\frac{n+2^2}{2^3}\right\rfloor+\cdots=n.$$

M25. Evaluate

$$\sum_{0\le i<j\le n}\left\lfloor\frac{x+i}{j}\right\rfloor,$$

where x is a real number.

M26. Let x, y, and z be integers such that $xy+xz+yz=0$. Prove that $(x+y+z)^2$ divides $x^5+y^5+z^5$.

M27. Let p be an odd prime. Prove that

$$\sum_{k=1}^{p-1} \frac{k^p - k}{p} \equiv \frac{p+1}{2} \pmod{p}.$$

M28. Prove that for each positive integer $n \geq 2$ the following inequality holds

$$\sigma(n)\phi(n) < n^2,$$

where $\phi(n)$ is the number of integers that are less than n and are relatively prime with n, and $\sigma(n)$ is the sum of the positive divisors of n.

M29. Let m and n be positive integers with m even and at least equal to 4. Prove that

$$\sum_{k=0}^{m} (-4)^k n^{4(m-k)}$$

is not a prime number.

M30. Let p be a prime such that $p \equiv 1 \pmod 3$ and let $q = \lfloor 2p/3 \rfloor$. If

$$\frac{1}{1 \cdot 2} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{(q-1)q} = \frac{m}{n}$$

for some integers m and n, prove that $p \mid m$.

M31. Prove that for different choices of the signs $+$ and $-$ the expression

$$\pm 1 \pm 2 \pm \cdots \pm (4n+1)$$

yields all odd positive integers less than or equal to $(2n+1)(4n+1)$.

M32. Let n be a positive integer. Prove that all binomial coefficients $\binom{n}{k}$ with $0 \leq k \leq n$ are odd if and only if $n = 2^m - 1$ for some nonnegative integer m.

M33. For each positive integer n define

$$a_n = \frac{(n+1)(n+2)\cdots(n+2010)}{2010!}.$$

Prove that there are infinitely many n such that a_n is an integer with no prime factors less than 2010.

M34. The numbers $a_1, a_2, \ldots, a_n > 0$ and $b_1 \geq b_2 \geq \cdots \geq b_n > 0$ satisfy

$$a_1 \geq b_1,\ a_1 + a_2 \geq b_1 + b_2, \ldots,\ a_1 + a_2 + \cdots + a_n \geq b_1 + b_2 + \cdots + b_n.$$

Prove that for every positive integer j,

$$a_1^j + a_2^j + \cdots + a_n^j \geq b_1^j + b_2^j + \cdots + b_n^j.$$

M35. Evaluate

$$\sum_{k=1}^{\infty}\sum_{l=k}^{\infty}\frac{1}{l!}.$$

M36. Evaluate

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{i!j!}{(i+j+1)!}.$$

M37. Prove the inequality

$$\sum_{k=1}^{\infty}\frac{1}{(k+1)\sqrt{k}} < 2.$$

M38. Remember the identity from the Example 3.5 and use it to prove that

$$\sum_{k=1}^{\infty}\frac{1}{k^2} = \frac{\pi^2}{6}.$$

M39. Evaluate

(a) $\displaystyle\sum_{k=1}^{\infty}\frac{1}{1^3+2^3+\cdots+k^3}.$

(b) $\displaystyle\sum_{k=1}^{\infty}\frac{(-1)^k}{1^3+2^3+\cdots+k^3}.$

M40. Let T be the set of all triples (a,b,c) of positive integers such that a,b,c are the lengths of the sides of some triangle. Evaluate

$$\sum_{(a,b,c)\in T} \frac{2^a}{3^b 5^c}.$$

M41. Prove that the inequality

$$\left(\frac{a_1}{a_2}\right)^{n-1} + \left(\frac{a_2}{a_3}\right)^{n-1} + \cdots + \left(\frac{a_n}{a_1}\right)^{n-1} \geq 2\frac{a_1 + a_2 + \cdots + a_n}{\sqrt[n]{a_1 a_2 \cdots a_n}} - n$$

holds for any positive real numbers $a_1, a_2, \ldots, a_n$.

3 Hard Problems

H1. Find all positive integers n for which

$$N = \left(1^4 + \frac{1}{4}\right)\left(2^4 + \frac{1}{4}\right)\cdots\left(n^4 + \frac{1}{4}\right)$$

is the square of a rational number.

H2. Let $a_0 \geq 2$ and $a_{n+1} = a_n^2 - a_n + 1$, $n \geq 0$. Prove that

$$\log_{a_0}(a_n - 1)\log_{a_1}(a_n - 1)\cdots\log_{a_{n-1}}(a_n - 1) \geq n^n$$

for all $n \geq 1$.

H3. Let a be a real number greater than 1. Evaluate

$$\sum_{k=1}^{\infty}(-1)^{k-1}\frac{2^{k-1}a^{2^{k-1}-1}}{a^{2^k} - a^{2^{k-1}} + 1}.$$

H4. For a nonnegative integer k, define $S_k(n) = 1^k + 2^k + \cdots + n^k$. Prove that

$$1 + \sum_{k=0}^{r-1}\binom{r}{k}S_k(n) = (n+1)^r.$$

H5. Find all positive integers n such that

$$n = \prod_{i=0}^{m}(a_i + 1),$$

where $\overline{a_m a_{m-1}\ldots a_0}$ is the decimal representation of n.

H6. Let

$$a_k = \frac{k}{(k-1)^{\frac{4}{3}} + k^{\frac{4}{3}} + (k+1)^{\frac{4}{3}}}.$$

Prove that $a_1 + a_2 + \cdots + a_{999} < 50$.

H7. For a fixed positive integer a define the sequence

$$a_n = \left\lfloor \left(a + \sqrt{a^2+1}\right)^n + \left(\frac{1}{2}\right)^n \right\rfloor, \quad n \geq 0.$$

Prove that

$$\sum_{n=1}^{\infty} \frac{1}{a_{n-1}a_{n+1}} = \frac{1}{8a^2}.$$

H8. Evaluate

$$\sum_{k=1}^{\infty} \frac{1}{2^k} \tan \frac{a}{2^k},$$

where $a \neq s\pi$, with s any integer.

H9. Let n be a positive integer. Prove that

$$\prod_{k=0}^{n-1} \left(2\sin^2 \frac{(k-1)\pi}{n} + 2\sin^2 \frac{(k+1)\pi}{n} - \sin^2 \frac{2\pi}{n}\right) = \left(1 - \cos^n \frac{2\pi}{n}\right)^2.$$

H10. Let m and n be integers greater than 1. Prove that

$$\sum_{k_1+k_2+\cdots+k_n=m, k_1,k_2,\ldots,k_n \geq 0} \frac{1}{k_1!k_2!\cdots k_n!} \cos\left((k_1+2k_2+\cdots+nk_n)\frac{2\pi}{n}\right) = 0.$$

H11. Let X be a set with n elements. Prove that

$$\sum_{Y,Z \subseteq X} |Y \cap Z| = n \cdot 4^{n-1}.$$

The sum is over all possible pairs (Y, Z) of subsets of X.

H12. Evaluate the sum

$$S_n = \binom{n}{1} - 3\binom{n}{3} + 5\binom{n}{5} - 7\binom{n}{7} + \cdots$$

H13. Prove that

$$\sum_{k\equiv 1 \pmod 3} \binom{n}{k} = \frac{1}{3}\left(2^n + 2\cos\left(\frac{(n-2)\pi}{3}\right)\right).$$

H14. Prove that

$$\sum_{k=0}^{m} \binom{n}{k}\binom{n}{m-k} = \binom{2n}{m}$$

for any nonnegative integers m and n.

H15. Let n be a positive integer. Prove the combinatorial identity

$$\sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k} = \sum_{k=0}^{n} 2^k \binom{n}{k}^2.$$

H16. Let n be a positive integer. Prove that

$$\sum_{k=1}^{n} \frac{(-1)^{k-1}}{k}\binom{n}{k} = 1 + \frac{1}{2} + \cdots + \frac{1}{n}.$$

H17. Let $(F_n)_{n\geq 0}$ be the Fibonacci sequence defined by

$$F_0 = 0,\ F_1 = 1 \text{ and } F_{n+2} = F_{n+1} + F_n \text{ for } n \geq 0.$$

Prove that

$$\sum_{k\geq 0} \binom{n-k}{k} = F_{n+1}.$$

Obviously, the sum lasts as long as the binomial coefficient is not 0 (that is, as long as $n - k \geq k$).

H18. Evaluate

$$\binom{n}{0} - \binom{n-1}{1} + \binom{n-2}{2} - \binom{n-3}{3} + \cdots.$$

H19. Partition the set of positive integers into $n \geq 1$ arithmetic progressions with first terms $a_1, a_2, \dots, a_n$ and common differences $d_1, d_2, \dots, d_n$ respectively. Prove that

$$\sum_{k=1}^{n} \frac{a_k}{d_k} = \frac{n+1}{2}.$$

H20. Let $a_1 \leq a_2 \leq \cdots \leq a_n$ and $b_1, b_2, \dots, b_n$ be positive real numbers such that

$$a_1 + a_2 + \cdots + a_k \geq b_1 + b_2 + \cdots + b_k \text{ for all } 1 \leq k \leq n.$$

Prove that $a_1 a_2 \cdots a_n \geq b_1 b_2 \cdots b_n$.

H21. Prove that Carleman's inequality, that is,

$$\sum_{k=1}^{\infty} \sqrt[k]{a_1 a_2 \cdots a_k} \leq e \sum_{k=1}^{\infty} a_k$$

holds for every positive real numbers $a_1, a_2, \dots$.

H22. Prove that

$$\sum_{k=1}^{n^2} \left\lfloor \sqrt{k} \right\rfloor = \frac{n(4n^2 - 3n + 5)}{6}.$$

H23. Let p and q be relatively prime odd natural numbers. Prove that

$$\sum_{k=1}^{\frac{p-1}{2}} \left\lfloor \frac{kq}{p} \right\rfloor + \sum_{l=1}^{\frac{q-1}{2}} \left\lfloor \frac{lp}{q} \right\rfloor = \frac{(p-1)(q-1)}{4}.$$

H24. Let p be an odd prime. Prove that

$$\sum_{k=1}^{p-1} \left\lfloor \frac{k^3}{p} \right\rfloor = \frac{(p-2)(p-1)(p+1)}{4}.$$

H25. Let p be an odd prime and let $f : \mathbb{Z}_+ \to \mathbb{R}$ be a function such that

(i) $\dfrac{f(k)}{p}$ is not an integer, for $k = 1, 2, \ldots, p-1$;

(ii) $f(k) + f(p-k)$ is an integer divisible by p, for $k = 1, 2, \ldots, p-1$.

Prove that

$$\sum_{k=1}^{p-1} \left\lfloor \frac{f(k)}{p} \right\rfloor = \frac{1}{p} \sum_{k=1}^{p-1} f(k) - \frac{p-1}{2}.$$

H26. If $p > 3$ is a prime number and x, y, and z are integers such that $x+y+z$ and $xy+xz+yz$ are both divisible by p, then $x^p+y^p+z^p$ and $x^py^p+x^pz^p+y^pz^p$ are divisible by p^2.

H27. Let p be an odd prime and let

$$S_q = \frac{1}{2\cdot 3\cdot 4} + \frac{1}{5\cdot 6\cdot 7} + \cdots + \frac{1}{q(q+1)(q+2)},$$

where $q = \dfrac{3p-5}{2}$. Assume that $\dfrac{1}{p} - 2S_q = \dfrac{m}{n}$, for some integers m and n. Prove that $m \equiv n (\text{mod } p)$.

H28. Let n be a positive integer, and let 2^r be the highest power of 2 dividing n. Prove that 2^{2r} is the highest power of 2 dividing the numerator of

$$1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2n-1}$$

when the sum is represented as a fraction in its lowest terms.

H29. Let $n \geq 2$ be a positive integer, with divisors $1 = d_1 < d_2 < \ldots < d_k = n$. Prove that $d_1d_2+d_2d_3+\cdots+d_{k-1}d_k$ is always less than n^2, and determine when it is a divisor of n^2.

H30. Prove that

$$\sum_{d|n} \frac{\mu(d)}{d} = \frac{\phi(n)}{n}.$$

H31. Prove that

$$\sum_{d|n} \sigma(d)\mu\left(\frac{n}{d}\right) = n.$$

H32. Prove that

$$\prod_{d|n} d^{\mu(d)} = \begin{cases} 1, & \text{if} \quad n \text{ is not a power of a prime} \\ \dfrac{1}{p}, & \text{if} \quad n = p^a, \text{ with } p \text{ prime.} \end{cases}$$

H33. Let a_n be a sequence of integers that satisfies

$$\sum_{d|n} a_d = 2^n \text{ for all } n \geq 1.$$

Prove that $n \mid a_n$ for all $n \geq 1$.

H34. Prove that

$$\sum_{n=1}^{\infty} \frac{\phi(n)}{2^n - 1} = 2.$$

H35. Let p be a positive prime, and let r be a positive integer. Consider the positive integers n and m such that $n \geq m > p^r - p^{r-1}$ and the integers $a_1, \ldots, a_n$. For any $0 \leq j \leq n$ denote by s_j and t_j the number of sums of the form $a_{i_1} + \cdots + a_{i_j}$, with $1 \leq i_1 < \cdots < i_j \leq n$ which are, and, respectively, which are not divisible by p (thus $s_0 = 1$, $t_0 = 0$). Prove that

$$S = \sum_{j=0}^{m} (-1)^j \binom{n-m+j}{j} s_{m-j} \equiv 0 \pmod{p^r}$$

and

$$T = \sum_{j=0}^{m} (-1)^j \binom{n-m+j}{j} t_{m-j} \equiv 0 \pmod{p^r}.$$

H36. Evaluate

$$\frac{11}{1 \cdot 2 \cdot 3}\left(\frac{4}{5}\right) + \frac{12}{2 \cdot 3 \cdot 4}\left(\frac{4}{5}\right)^2 + \frac{13}{3 \cdot 4 \cdot 5}\left(\frac{4}{5}\right)^3 + \cdots$$

H37. Let a be a real number. Define the sequence $(x_n)_{n\geq 1}$ recursively by

$$x_1 = 1 \text{ and } x_{n+1} = a^n + nx_n, \text{ for } n \geq 1.$$

Prove that

$$\prod_{n=1}^{\infty}\left(1 - \frac{a^n}{x_{n+1}}\right) = e^{-a}.$$

H38. Evaluate

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{i!j!}{(i+j+2)!}.$$

H39. Prove that

$$\sum_{k=1}^{\infty}\frac{1}{k^4} = \frac{\pi^4}{90}.$$

H40. Evaluate

$$\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}\sum_{l=0}^{\infty}\frac{1}{k2^l+1}.$$

H41. Let $a_1, a_2, \ldots, a_{100}$ be nonnegative real numbers such that

$$a_1^2 + a_2^2 + \cdots + a_{100}^2 = 1.$$

Prove that

$$a_1^2a_2 + a_2^2a_3 + \cdots + a_{100}^2a_1 \leq \frac{\sqrt{2}}{3}.$$

H42. Let $x_1, \ldots, x_{100}$ be nonnegative real numbers such that

$$x_i + x_{i+1} + x_{i+2} \leq 1 \text{ for all } i = 1, \ldots, 100$$

(set $x_{101} = x_1, x_{102} = x_2$). Find the maximal possible value of the sum

$$S = \sum_{i=1}^{100} x_i x_{i+2}.$$

H43. Prove that for any real numbers $x_1, x_2, \ldots, x_n$ and any nonnegative real numbers $r_1, r_2, \ldots, r_n$ the inequality

$$\sum_{i,j=1}^{n} \min(r_i, r_j) x_i x_j \geq 0$$

holds. (The sum is over all pairs (i, j) with $1 \leq i \leq n$ and $1 \leq j \leq n$.)

H44. Let $a_1, b_1, a_2, b_2, \ldots, a_n, b_n$ be nonnegative real numbers. Prove that

$$\sum_{i,j=1}^{n} \min(a_i a_j, b_i b_j) \leq \sum_{i,j=1}^{n} \min(a_i b_j, a_j b_i).$$

Chapter 9

Solutions

1 Solutions to Easy Problems

E1. *Evaluate*

$$\frac{(6!+5!)(5!+4!)(4!+3!)(3!+2!)(2!+1!)}{(6!-5!)(5!-4!)(4!-3!)(3!-2!)(2!-1!)}.$$

Solution. We have

$$\frac{(6!+5!)(5!+4!)(4!+3!)(3!+2!)(2!+1!)}{(6!-5!)(5!-4!)(4!-3!)(3!-2!)(2!-1!)}$$
$$= \frac{(7\cdot 5!)(6\cdot 4!)(5\cdot 3!)(4\cdot 2!)(3\cdot 1!)}{(5\cdot 5!)(4\cdot 4!)(3\cdot 3!)(2\cdot 2!)(1\cdot 1!)} = 21,$$

after simplifying. We used the formulae

$$(k+1)!+k! = (k+2)\cdot k! \quad \text{or} \quad (k+1)!-k! = k\cdot k!.$$

E2. *Prove that*

$$1\cdot n + 2\cdot(n-1) + 3\cdot(n-2) + \cdots + (n-1)\cdot 2 + n\cdot 1 = \frac{n(n+1)(n+2)}{6}.$$

Solution. Indeed, we have

$$1\cdot n + 2\cdot(n-1) + 3\cdot(n-2) + \cdots + (n-1)\cdot 2 + n\cdot 1 = \sum_{k=1}^{n} k(n+1-k)$$

$$= (n+1)\sum_{k=1}^{n} k - \sum_{k=1}^{n} k^2 = (n+1)\frac{n(n+1)}{2} - \frac{n(n+1)(2n+1)}{6}$$
$$= \frac{n(n+1)}{6}(3n+3-2n-1) = \frac{n(n+1)(n+2)}{6},$$

according to the well-known formulae for the sum of the first n positive integers and the sum of their squares.

E3. *Find a general formula for the sum*

$$1 + 11 + 111 + \cdots + \underbrace{11\ldots11}_{n \text{ digits}}.$$

Solution. We have

$$1 + 11 + 111 + \cdots + \underbrace{11\ldots11}_{n \text{ digits}} = \sum_{k=1}^{n} \frac{10^k - 1}{9} = \frac{10}{9}\sum_{k=1}^{n} 10^{k-1} - \frac{n}{9}$$
$$= \frac{10}{9} \cdot \frac{10^n - 1}{9} - \frac{n}{9}$$
$$= \frac{1}{81}\left(10^{n+1} - 9n - 10\right).$$

We used the well-known formula for the sum of a geometric progression:

$$\sum_{k=1}^{n} q^{k-1} = \sum_{j=0}^{n-1} q^j = \frac{q^n - 1}{q - 1},$$

for all $q \neq 1$.

E4. *Evaluate the product*

$$\prod_{k=2}^{n} \frac{4k^3 - 3k + 1}{4k^3 - 3k - 1}.$$

Solution. It is easy to factor

$$4k^3 - 3k + 1 = (k+1)(2k-1)^2$$

and

$$4k^3 - 3k - 1 = (k-1)(2k+1)^2.$$

Thus the product that we want to calculate becomes

$$\begin{aligned}\prod_{k=2}^{n} \frac{4k^3 - 3k + 1}{4k^3 - 3k - 1} &= \prod_{k=2}^{n} \left(\frac{k+1}{k-1}\right) \prod_{k=2}^{n} \left(\frac{2k-1}{2k+1}\right)^2 \\ &= \prod_{k=2}^{n} \left(\frac{k+1}{k-1}\right) \left(\prod_{k=2}^{n} \frac{2k-1}{2k+1}\right)^2 \\ &= \frac{n(n+1)}{1 \cdot 2} \left(\frac{3}{2n+1}\right)^2 = \frac{9n(n+1)}{2(2n+1)^2}\end{aligned}$$

since both products telescope.

E5. *For a fixed positive integer n let $a_k = 2^{2^{k-n}} + k$, $k = 0, 1, \ldots, n$. Prove that*

$$(a_1 - a_0) \cdots (a_n - a_{n-1}) = \frac{7}{a_1 + a_0}.$$

Solution. Here we use

$$(a^2 + a + 1)(a^2 - a + 1) = (a^2 + 1)^2 - a^2 = a^4 + a^2 + 1,$$

more precisely we use it in the form

$$a^2 - a + 1 = \frac{a^4 + a^2 + 1}{a^2 + a + 1}.$$

This allows telescoping products like

$$(a^2 - a + 1)(a^4 - a^2 + 1) \cdots (a^{2^n} - a^{2^{n-1}} + 1)$$

$$= \frac{a^4 + a^2 + 1}{a^2 + a + 1} \cdot \frac{a^8 + a^4 + 1}{a^4 + a^2 + 1} \cdots \frac{a^{2^{n+1}} + a^{2^n} + 1}{a^{2^n} + a^{2^{n-1}} + 1} = \frac{a^{2^{n+1}} + a^{2^n} + 1}{a^2 + a + 1}.$$

In our case

$$\prod_{k=1}^{n}(a_k - a_{k-1}) = \prod_{k=1}^{n}(2^{2^{k-n}} - 2^{2^{k-1-n}} + 1) = \prod_{k=1}^{n} \frac{2^{2^{k+1-n}} + 2^{2^{k-n}} + 1}{2^{2^{k-n}} + 2^{2^{k-1-n}} + 1}$$
$$= \frac{2^{2^{n+1-n}} + 2^{2^{n-n}} + 1}{2^{2^{1-n}} + 2^{2^{1-1-n}} + 1} = \frac{7}{2^{2^{1-n}} + 1 + 2^{2^{-n}}} = \frac{7}{a_1 + a_0}.$$

E6. *Prove that for all positive integers n*

$$\sum_{k=1}^{n} \frac{k}{(k+1)!} < 1.$$

Solution. We have, indeed,

$$\sum_{k=1}^{n} \frac{k}{(k+1)!} = \sum_{k=1}^{n} \left(\frac{k+1}{(k+1)!} - \frac{1}{(k+1)!} \right)$$
$$= \sum_{k=1}^{n} \left(\frac{1}{k!} - \frac{1}{(k+1)!} \right)$$
$$= 1 - \frac{1}{(n+1)!} < 1.$$

E7. *Evaluate*

$$\sum_{k=1}^{n} \frac{k+1}{(k-1)! + k! + (k+1)!}.$$

Solution. We have

$$(k-1)! + k! + (k+1)! = (k-1)!(1 + k + k(k+1)) = (k-1)!(k+1)^2,$$

hence

$$\sum_{k=1}^{n} \frac{k+1}{(k-1)! + k! + (k+1)!} = \sum_{k=1}^{n} \frac{1}{(k-1)!(k+1)}$$
$$= \sum_{k=1}^{n} \frac{k}{(k+1)!} = 1 - \frac{1}{(n+1)!},$$

as we have seen in the previous problem.

E8. *Evaluate*

$$\prod_{k=1}^{n}\left(\frac{1}{8}+\frac{k+1}{(2k+1)^2}\right).$$

Solution. Because

$$\frac{1}{8}+\frac{k+1}{(2k+1)^2}=\frac{4k^2+12k+9}{8(2k+1)^2}=\frac{1}{8}\cdot\frac{(2k+3)^2}{(2k+1)^2}=\frac{1}{8}\cdot\frac{(2(k+1)+1)^2}{(2k+1)^2},$$

the product telescopes, as expected:

$$\prod_{k=1}^{n}\left(\frac{1}{8}+\frac{k+1}{(2k+1)^2}\right)=\frac{1}{8^n}\prod_{k=1}^{n}\frac{(2k+3)^2}{(2k+1)^2}=\frac{(2n+3)^2}{8^n\cdot 9}.$$

E9. *Evaluate*

$$\sum_{n\geq 2}\frac{3n^2+1}{(n^3-n)^3}.$$

Solution. Notice that

$$2(3n^2+1)=(n+1)^3-(n-1)^3 \quad\text{and}\quad n^3-n=n(n-1)(n+1),$$

therefore

$$\begin{aligned}\sum_{n=2}^{N}\frac{3n^2+1}{(n^3-n)^3}&=\frac{1}{2}\sum_{n=2}^{N}\frac{2(3n^2+1)}{(n^3-n)^3}=\frac{1}{2}\sum_{n=2}^{N}\left(\frac{1}{(n(n-1))^3}-\frac{1}{((n+1)n)^3}\right)\\&=\frac{1}{16}-\frac{1}{2}\cdot\frac{1}{((N+1)N)^3},\end{aligned}$$

for any positive integer $N\geq 2$. Consequently,

$$\sum_{n\geq 2}\frac{3n^2+1}{(n^3-n)^3}=\lim_{N\to\infty}\left(\frac{1}{16}-\frac{1}{2}\cdot\frac{1}{((N+1)N)^3}\right)=\frac{1}{16}.$$

E10. *Evaluate*

$$\sum_{k=1}^{n} k!(k^2+1).$$

Solution. We have

$$\begin{aligned}\sum_{k=1}^{n} k!(k^2+1) &= \sum_{k=1}^{n} k!\left((k+1)^2-2(k+1)+2\right)\\ &= \sum_{k=1}^{n}((k+1)!(k+1)-2(k+1)!+2k!)\\ &= \sum_{k=1}^{n}((k+2)!-(k+1)!)-2\sum_{k=1}^{n}((k+1)!-k!)\\ &= (n+2)!-2!-2\left((n+1)!-1!\right)\\ &= (n+2)!-2(n+1)! = n(n+1)!.\end{aligned}$$

We used

$$(k+1)!(k+1) = (k+1)!((k+2)-1) = (k+2)!-(k+1)!.$$

E11. *Consider n arithmetic progressions with the same common difference d and having their first terms $1,2,3,\dots,n$. If $S(n,k)$ is the sum of the first n terms of the arithmetic progression that has its first term k, prove that*

$$S(n,1)+S(n,2)+S(n,3)+\cdots+S(n,n) = \frac{n^2}{2}(2+(n-1)(d+1)).$$

Solution. Indeed, we have that

$$S(n,k) = \frac{n(2k+(n-1)d)}{2},$$

for each $1 \le k \le n$, therefore

$$\begin{aligned}\sum_{k=1}^{n} S(n,k) &= \sum_{k=1}^{n}\left(nk+\frac{n(n-1)}{2}d\right) = n\sum_{k=1}^{n} k + n\frac{n(n-1)}{2}d\\ &= \frac{n^2(n+1)}{2}+\frac{n^2(n-1)}{2}d = \frac{n^2}{2}\left(2+(n-1)(d+1)\right).\end{aligned}$$

E12. *Let $a_1, a_2, \ldots, a_n$ be an arithmetic progression with $a_k \neq 0$ for all $1 \leq k \leq n$. Prove that*

$$\sum_{k=1}^{n-1} \frac{1}{a_k a_{k+1}} = \frac{n-1}{a_1 a_n}.$$

Solution. Let d be the common difference of the progression, that is,

$$d = a_2 - a_1 = \cdots = a_n - a_{n-1},$$

and, more general, $a_j - a_i = (j-i)d$ for all $i, j \in \{1, \ldots, n\}$.

The key observation (for telescoping) is that

$$\frac{1}{a_k a_{k+1}} = \frac{1}{d} \cdot \frac{d}{a_k a_{k+1}} = \frac{1}{d} \cdot \frac{a_{k+1} - a_k}{a_k a_{k+1}} = \frac{1}{d}\left(\frac{1}{a_k} - \frac{1}{a_{k+1}}\right).$$

Therefore

$$\begin{aligned}\sum_{k=1}^{n-1} \frac{1}{a_k a_{k+1}} &= \frac{1}{d}\sum_{k=1}^{n-1}\left(\frac{1}{a_k} - \frac{1}{a_{k+1}}\right) = \frac{1}{d}\left(\frac{1}{a_1} - \frac{1}{a_n}\right) \\ &= \frac{1}{d} \cdot \frac{a_n - a_1}{a_1 a_n} = \frac{n-1}{a_1 a_n},\end{aligned}$$

as desired.

Many sums of this type are encountered; for instance,

$$\sum_{k=1}^{n-1} \frac{1}{k(k+1)} = \sum_{k=1}^{n-1}\left(\frac{1}{k} - \frac{1}{k+1}\right) = 1 - \frac{1}{n} = \frac{n-1}{n},$$

or

$$\begin{aligned}\sum_{k=1}^{n-1} \frac{1}{(6k-1)(6k+5)} &= \frac{1}{6}\sum_{k=1}^{n-1}\left(\frac{1}{6k-1} - \frac{1}{6k+5}\right) \\ &= \frac{1}{6}\left(\frac{1}{5} - \frac{1}{6n-1}\right) = \frac{n-1}{5(6n-1)}.\end{aligned}$$

This also allows the computation of the corresponding infinite series. Thus

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)} = \lim_{n\to\infty} \sum_{k=1}^{n-1} \frac{1}{k(k+1)} = \lim_{n\to\infty} \left(1 - \frac{1}{n}\right) = 1,$$

and, similarly

$$\sum_{k=1}^{\infty} \frac{1}{(6k-1)(6k+5)} = \lim_{n\to\infty} \sum_{k=1}^{n-1} \frac{1}{(6k-1)(6k+5)}$$
$$= \lim_{n\to\infty} \frac{1}{6}\left(\frac{1}{5} - \frac{1}{6n-1}\right) = \frac{1}{30}.$$

Notice that a converse is also true. Namely, if, for the nonzero real numbers $a_1, \ldots, a_n$, we have

$$\sum_{k=1}^{j-1} \frac{1}{a_k a_{k+1}} = \frac{j-1}{a_1 a_j}$$

for all $2 \le j \le n$, then $a_1, \ldots, a_n$ form an arithmetic progression. Indeed, by subtracting the above equation from the similar one obtained by replacing j to $j+1$, that is from

$$\sum_{k=1}^{j} \frac{1}{a_k a_{k+1}} = \frac{j}{a_1 a_{j+1}},$$

we obtain

$$\frac{1}{a_j a_{j+1}} = \frac{j}{a_1 a_{j+1}} - \frac{j-1}{a_1 a_j}.$$

After clearing the denominators and rearranging a bit, we get

$$(j-1)(a_{j+1} - a_j) = a_j - a_1$$

for $2 \le j \le n-1$. For example, when $j = 2$, this gives $a_3 - a_2 = a_2 - a_1$. We denote $d = a_3 - a_2 = a_2 - a_1$, and see that $a_k = a_1 + (k-1)d$ is true

for $k \in \{1, 2, 3\}$. Assuming this to be true for $k = j$, we get (from the above equality)

$$(j-1)(a_{j+1} - a_j) = (j-1)d \Rightarrow a_{j+1} = a_j + d \Rightarrow a_{j+1} = a_1 + jd,$$

thus $a_k = a_1 + (k-1)d$ is shown to be true for all $k \le n$, by induction – which is what we intended to prove.

E13. *Let $a_1, a_2, \ldots, a_n$ be an arithmetic progression with positive terms. Prove that*

$$\frac{1}{\sqrt{a_1} + \sqrt{a_2}} + \frac{1}{\sqrt{a_2} + \sqrt{a_3}} + \cdots + \frac{1}{\sqrt{a_{n-1}} + \sqrt{a_n}} = \frac{n-1}{\sqrt{a_1} + \sqrt{a_n}}.$$

Solution. Let d be the common difference of the progression, so that

$$a_{k+1} - a_k = d \text{ for } 1 \le k \le n-1.$$

Then

$$\begin{aligned}
\sum_{k=1}^{n-1} \frac{1}{\sqrt{a_k} + \sqrt{a_{k+1}}} &= \sum_{k=1}^{n-1} \frac{\sqrt{a_{k+1}} - \sqrt{a_k}}{a_{k+1} - a_k} = \frac{1}{d} \sum_{k=1}^{n-1} \left(\sqrt{a_{k+1}} - \sqrt{a_k}\right) \\
&= (n-1)\frac{\sqrt{a_n} - \sqrt{a_1}}{(n-1)d} = (n-1)\frac{\sqrt{a_n} - \sqrt{a_1}}{a_n - a_1} \\
&= \frac{n-1}{\sqrt{a_1} + \sqrt{a_n}}.
\end{aligned}$$

Prove in an analogous manner that

$$\sum_{k=1}^{n-1} \frac{1}{\sqrt[3]{a_k^2} + \sqrt[3]{a_k a_{k+1}} + \sqrt[3]{a_{k+1}^2}} = \frac{n-1}{\sqrt[3]{a_1^2} + \sqrt[3]{a_1 a_n} + \sqrt[3]{a_n^2}}.$$

E14. *Prove the inequality*

$$\frac{1}{\sqrt{1} + \sqrt{3}} + \frac{1}{\sqrt{5} + \sqrt{7}} + \cdots + \frac{1}{\sqrt{9997} + \sqrt{9999}} > 24.$$

Solution. We proceed as in the previous problem in order to get

$$\sum_{k=1}^{5000} \frac{1}{\sqrt{2k-1}+\sqrt{2k+1}} = \sum_{k=1}^{5000} \frac{\sqrt{2k+1}-\sqrt{2k-1}}{2}$$
$$= \frac{1}{2}(\sqrt{10001}-1) > \frac{99}{2},$$

the inequality being equivalent to $\sqrt{10001} > 100 \Leftrightarrow 10001 > 10000$. Now, for the sum from the statement of the problem we have

$$\sum_{j=1}^{2500} \frac{1}{\sqrt{4j-3}+\sqrt{4j-1}} > \frac{1}{2} \sum_{j=1}^{2500} \left(\frac{1}{\sqrt{4j-3}+\sqrt{4j-1}} + \frac{1}{\sqrt{4j-1}+\sqrt{4j+1}} \right)$$
$$= \frac{1}{2} \sum_{k=1}^{5000} \frac{1}{\sqrt{2k-1}+\sqrt{2k+1}} > \frac{99}{4} > 24,$$

as desired.

E15. *Evaluate*

$$\sum_{k=1}^{n} \frac{k^2 - \frac{1}{2}}{k^4 + \frac{1}{4}}.$$

Solution. We have

$$\frac{k^2-\frac{1}{2}}{k^4+\frac{1}{4}} = \frac{\left(k-\frac{1}{2}\right)\left(k^2+k+\frac{1}{2}\right) - \left(k+\frac{1}{2}\right)\left(k^2-k+\frac{1}{2}\right)}{\left(k^2+k+\frac{1}{2}\right)\left(k^2-k+\frac{1}{2}\right)}$$
$$= \frac{k-\frac{1}{2}}{k^2-k+\frac{1}{2}} - \frac{k+\frac{1}{2}}{k^2+k+\frac{1}{2}}$$
$$= \frac{k-\frac{1}{2}}{k^2-l+\frac{1}{2}} - \frac{(k+1)-\frac{1}{2}}{(k+1)^2-(k+1)+\frac{1}{2}},$$

consequently

$$\sum_{k=1}^{n} \frac{k^2 - \frac{1}{2}}{k^4 + \frac{1}{4}} = \sum_{k=1}^{n} \left(\frac{k - \frac{1}{2}}{k^2 - k + \frac{1}{2}} - \frac{(k+1) - \frac{1}{2}}{(k+1)^2 - (k+1) + \frac{1}{2}} \right)$$
$$= \frac{1 - \frac{1}{2}}{1^2 - 1 + \frac{1}{2}} - \frac{(n+1) - \frac{1}{2}}{(n+1)^2 - (n+1) + \frac{1}{2}}$$
$$= 1 - \frac{n + \frac{1}{2}}{n^2 + n + \frac{1}{2}} = \frac{n^2}{n^2 + n + \frac{1}{2}}.$$

Note that this implies

$$\sum_{k=1}^{\infty} \frac{k^2 - \frac{1}{2}}{k^4 + \frac{1}{4}} = 1.$$

E16. *Prove that*

$$\sum_{k=1}^{9999} \frac{1}{\left(\sqrt{k} + \sqrt{k+1}\right)\left(\sqrt[4]{k} + \sqrt[4]{k+1}\right)} = 9.$$

Solution. Indeed, we have

$$\left(\sqrt{a} + \sqrt{b}\right)\left(\sqrt[4]{a} + \sqrt[4]{b}\right)\left(\sqrt[4]{a} - \sqrt[4]{b}\right) = \left(\sqrt{a} + \sqrt{b}\right)\left(\sqrt{a} - \sqrt{b}\right) = a - b$$

for nonnegative a and b, thus

$$\left(\sqrt{k+1} + \sqrt{k}\right)\left(\sqrt[4]{k+1} + \sqrt[4]{k}\right)\left(\sqrt[4]{k+1} - \sqrt[4]{k}\right) = 1$$

for $k \geq 0$, and

$$\sum_{k=1}^{9999} \frac{1}{\left(\sqrt{k} + \sqrt{k+1}\right)\left(\sqrt[4]{k} + \sqrt[4]{k+1}\right)} = \sum_{k=1}^{9999} \left(\sqrt[4]{k+1} - \sqrt[4]{k}\right)$$
$$= \sqrt[4]{10000} - 1 = 9.$$

E17. *Find the closed form*

$$\sqrt{1+\frac{1}{1^2}+\frac{1}{2^2}}+\sqrt{1+\frac{1}{2^2}+\frac{1}{3^2}}+\cdots+\sqrt{1+\frac{1}{1999^2}+\frac{1}{2000^2}}.$$

Solution. Here we must observe that

$$1+\frac{1}{k^2}+\frac{1}{(k+1)^2}=\frac{k^4+2k^3+3k^2+2k+1}{k^2(k+1)^2}=\frac{(k^2+k+1)^2}{k^2(k+1)^2}$$
$$=\left(1+\frac{1}{k(k+1)}\right)^2.$$

Consequently,

$$\sum_{k=1}^{1999}\sqrt{1+\frac{1}{k^2}+\frac{1}{(k+1)^2}}=\sum_{k=1}^{1999}\left(1+\frac{1}{k(k+1)}\right)=\sum_{k=1}^{1999}\left(1+\frac{1}{k}-\frac{1}{k+1}\right)$$
$$=1999+1-\frac{1}{2000}=\frac{3999999}{2000}.$$

E18. *Evaluate the sum*

$$\sum_{k=1}^{n}\frac{1}{\sqrt{2k+\sqrt{4k^2-1}}}.$$

Solution. Knowing the formulas

$$\sqrt{a\pm\sqrt{b}}=\sqrt{\frac{a+\sqrt{a^2-b}}{2}}\pm\sqrt{\frac{a-\sqrt{a^2-b}}{2}}$$

for de-nesting nested radicals of order 2 can be very helpful here. (Of course these formulas work for nonnegative a and b such that $a^2-b\geq 0$ and they indeed de-nest the radicals only if a^2-b is a square.) For $a=2k$ and $b=4k^2-1$ (and with a plus sign) we find

$$\sqrt{2k+\sqrt{4k^2-1}}=\sqrt{\frac{2k+1}{2}}+\sqrt{\frac{2k-1}{2}},$$

and reveal the sum to be an old friend:

$$\begin{aligned}\sum_{k=1}^{n}\frac{1}{\sqrt{2k+\sqrt{4k^2-1}}} &= \sqrt{2}\sum_{k=1}^{n}\frac{1}{\sqrt{2k+1}+\sqrt{2k-1}}\\ &= \frac{\sqrt{2}}{2}\sum_{k=1}^{n}\left(\sqrt{2k+1}-\sqrt{2k-1}\right)\\ &= \frac{\sqrt{2}}{2}(\sqrt{2n+1}-1).\end{aligned}$$

E19. *Let* $a_n = \sqrt{1+\left(1+\frac{1}{n}\right)^2}+\sqrt{1+\left(1-\frac{1}{n}\right)^2}$, $n \geq 1$. *Prove that*

$$\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_{20}}$$

is a positive integer.

Solution. We have

$$\frac{1}{a_k}=\frac{k}{\sqrt{2k^2+2k+1}+\sqrt{2k^2-2k+1}}=\frac{\sqrt{2k^2+2k+1}-\sqrt{2k^2-2k+1}}{4},$$

therefore

$$\begin{aligned}\sum_{k=1}^{20}\frac{1}{a_k} &= \frac{1}{4}\sum_{k=1}^{20}\left(\sqrt{2k^2+2k+1}-\sqrt{2k^2-2k+1}\right)\\ &= \frac{1}{4}\sum_{k=1}^{20}\left(\sqrt{2(k+1)^2-2(k+1)+1}-\sqrt{2k^2-2k+1}\right)\\ &= \frac{1}{4}\left(\sqrt{2\cdot 21^2-2\cdot 21+1}-\sqrt{2\cdot 1^2-2\cdot 1+1}\right)=7,\end{aligned}$$

a positive integer.

E20. *Prove that*

$$\sum_{k=0}^{n}\frac{1}{k!(n-k)!}=\frac{2^n}{n!}.$$

Solution. This is more like a joke: after multiplying by $n!$, the identity to prove reads

$$\sum_{k=0}^{n} \frac{n!}{k!(n-k)!} = 2^n,$$

that is,

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n,$$

which follows instantly from the binomial theorem.

E21. *Evaluate*

$$\frac{\binom{n}{0}}{1} + \frac{\binom{n}{1}}{2} + \cdots + \frac{\binom{n}{n}}{n+1}.$$

Solution 1. We have

$$\begin{aligned}\frac{1}{k+1}\binom{n}{k} &= \frac{1}{k+1} \cdot \frac{n!}{k!(n-k)!} = \frac{1}{n+1} \cdot \frac{(n+1)!}{(k+1)!(n-k)!} \\ &= \frac{1}{n+1}\binom{n+1}{k+1},\end{aligned}$$

therefore

$$\begin{aligned}\sum_{k=0}^{n} \frac{1}{k+1}\binom{n}{k} &= \sum_{k=0}^{n} \frac{1}{n+1}\binom{n+1}{k+1} = \frac{1}{n+1}\sum_{j=1}^{n+1}\binom{n+1}{j} \\ &= \frac{1}{n+1}(2^{n+1}-1),\end{aligned}$$

by the fundamental formula, again. Indeed, we have

$$\sum_{j=1}^{n+1}\binom{n+1}{j} = 2^{n+1} - \binom{n+1}{0} = 2^{n+1} - 1.$$

Solution 2. Now we integrate the formula

$$\sum_{k=0}^{n} \binom{n}{k} x^k = (1+x)^n$$

with respect to x, on the interval $[0,t]$. We get

$$\int_0^t \left(\sum_{k=0}^{n}\binom{n}{k}x^k\right)dx = \int_0^t (1+x)^n dx$$

$$\Leftrightarrow \sum_{k=0}^{n}\binom{n}{k}\frac{t^{k+1}}{k+1} = \frac{1}{n+1}\left((1+t)^n - 1\right).$$

Again we arrive at a more general formula than the one that we need to prove, which can be obtained by specializing to $t=1$.

E22. *Prove that*

$$\binom{n}{m}+\binom{n-1}{m-1}+\binom{n-2}{m-2}+\cdots+\binom{n-m}{0}=\binom{n+1}{m}.$$

Solution. By the recurrence formula of the binomial coefficients, that is

$$\binom{a+1}{b}=\binom{a}{b}+\binom{a}{b-1}\Leftrightarrow\binom{a}{b}=\binom{a+1}{b}-\binom{a}{b-1},$$

we have

$$\begin{aligned}\sum_{j=0}^{m}\binom{n-j}{m-j} &= \sum_{j=0}^{m}\left(\binom{n-j+1}{m-j}-\binom{n-j}{m-j-1}\right)\\ &= \sum_{j=0}^{m}\left(\binom{n-j+1}{m-j}-\binom{n-(j+1)+1}{m-(j+1)}\right)\\ &= \binom{n+1}{m}.\end{aligned}$$

Note that, when used for $j=m$, the recurrence formula is usually written as

$$\binom{n-m}{0}=\binom{n-m+1}{0}$$

rather than

$$\binom{n-m}{0}=\binom{n-m+1}{0}-\binom{n-m}{-1}$$

(the last binomial coefficient is considered to be 0). Also note that we can also put the formula in the form

$$\binom{n}{n-m}+\binom{n-1}{n-m}+\binom{n-2}{n-m}+\cdots+\binom{n-m}{n-m}=\binom{n+1}{n-m+1}.$$

(due to the formula $\binom{a}{b}=\binom{a}{a-b}$), and, if we denote $p=n-m$, it becomes

$$\binom{n}{p}+\binom{n-1}{p}+\binom{n-2}{p}+\cdots+\binom{p}{p}=\binom{n+1}{p+1},\quad n\geq p.$$

We have here basically the same identity, but in a different form, that often appears in the literature. After multiplying by $p!$, we can also express this as

$$\sum_{j=p}^{n} j(j-1)\cdots(j-p+1)=\frac{(n+1)n\cdots(n-p+1)}{p+1}.$$

This last identity (which we met in the first chapter) can be proved by telescoping with

$$\begin{aligned}j(j-1)\cdots(j-p+1)=\frac{1}{p+1}&((j+1)j(j-1)\cdots(j-p+1)\\&-j(j-1)\cdots(j-p+1)(j-p)).\end{aligned}$$

E23. *Evaluate*

$$\binom{2008}{3}-2\binom{2008}{4}+3\binom{2008}{5}-4\binom{2008}{6}+\cdots-2004\binom{2008}{2006}+2005\binom{2008}{2007}.$$

Solution. The two already discussed formulae

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}=0\quad\text{and}\quad\sum_{k=0}^{n}(-1)^k k\binom{n}{k}=0$$

which hold for $n \geq 2$ imply

$$\sum_{k=0}^{n}(-1)^k(k-2)\binom{n}{k} = \sum_{k=0}^{n}(-1)^k k\binom{n}{k} - 2\sum_{k=0}^{n}(-1)^k\binom{n}{k} = 0.$$

We can rewrite this as

$$-2\binom{n}{0} + \binom{n}{1} + \sum_{k=3}^{n-1}(-1)^k(k-2)\binom{n}{k} + (-1)^n(n-2)\binom{n}{2} = 0,$$

and we deduce from this one that, when $n \geq 4$,

$$\sum_{k=3}^{n-1}(-1)^{k-1}(k-2)\binom{n}{k} = -2+n+(-1)^n(n-2) = (n-2)(1+(-1)^n).$$

For $n = 2008$ we get

$$\sum_{k=3}^{2007}(-1)^{k-1}(k-2)\binom{n}{k} = 4012,$$

and this is the sum we are asked for.

E24. *Evaluate the sum*

$$\sum_{k=1}^{\infty}\frac{6^k}{(3^k-2^k)(3^{k+1}-2^{k+1})}.$$

Solution. We have

$$\begin{aligned}
\sum_{k=1}^{\infty}\frac{6^k}{(3^k-2^k)(3^{k+1}-2^{k+1})} &= \lim_{n\to\infty}\sum_{k=1}^{n}\frac{6^k}{(3^k-2^k)(3^{k+1}-2^{k+1})}\\
&= \lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{2^k}{3^k-2^k}-\frac{2^{k+1}}{3^{k+1}-2^{k+1}}\right)\\
&= \lim_{n\to\infty}\left(2-\frac{2^{n+1}}{3^{n+1}-2^{n+1}}\right)\\
&= \lim_{n\to\infty}\left(2-\frac{(2/3)^{n+1}}{1-(2/3)^{n+1}}\right) = 2.
\end{aligned}$$

E25. *Let F_n be the n^{th} Fibonacci number ($F_1 = F_2 = 1$, and $F_n = F_{n-1}+F_{n-2}$ for all $n \geq 3$). Evaluate*

$$\sum_{k=2}^{\infty} \frac{F_k}{F_{k-1}F_{k+1}}.$$

Solution. We have

$$\begin{aligned}
\sum_{k=2}^{\infty} \frac{F_k}{F_{k-1}F_{k+1}} &= \lim_{n\to\infty} \sum_{k=2}^{n} \frac{F_k}{F_{k-1}F_{k+1}} = \lim_{n\to\infty} \sum_{k=2}^{n} \frac{F_{k+1}-F_{k-1}}{F_{k-1}F_{k+1}} \\
&= \lim_{n\to\infty} \sum_{k=2}^{n} \left(\frac{1}{F_{k-1}} - \frac{1}{F_{k+1}}\right) \\
&= \lim_{n\to\infty} \left(\frac{1}{F_1} + \frac{1}{F_2} - \frac{1}{F_n} - \frac{1}{F_{n+1}}\right) \\
&= \frac{1}{F_1} + \frac{1}{F_2} = 2,
\end{aligned}$$

since $\lim\limits_{n\to\infty} \dfrac{1}{F_n}$ clearly equals 0.

E26. *Prove that for all $n \geq 3$,*

$$\prod_{k=2}^{n-1} \left(\frac{1}{9} + \frac{k^2+k+1}{(k-1)^3}\right) = \frac{1}{3^{2n-1}} \left(\frac{n^3-n}{2}\right)^3.$$

Solution. We have

$$\begin{aligned}
\frac{1}{9} + \frac{k^2+k+1}{(k-1)^3} &= \frac{(k-1)^3 + 9(k^2+k+1)}{9(k-1)^3} = \frac{k^3+6k^2+12k+8}{9(k-1)^3} \\
&= \frac{(k+2)^3}{9(k-1)^3},
\end{aligned}$$

thus

$$\begin{aligned}
\prod_{k=2}^{n-1} \left(\frac{1}{9} + \frac{k^2+k+1}{(k-1)^3}\right) &= \prod_{k=2}^{n-1} \frac{(k+2)^3}{9(k-1)^3} = \frac{(n-1)^3 n^3 (n+1)^3}{9^{n-2} \cdot 1^3 \cdot 2^3 \cdot 3^3} \\
&= \frac{1}{3^{2n-1}} \left(\frac{n^3-n}{2}\right)^3.
\end{aligned}$$

Or, because the result is already given, we can use induction. For the induction step we need to show that

$$\frac{1}{3^{2n-1}}\left(\frac{n^3-n}{2}\right)^3\left(\frac{1}{9}+\frac{n^2+n+1}{(n-1)^3}\right)=\frac{1}{3^{2(n+1)-1}}\left(\frac{(n+1)^3-(n+1)}{2}\right)^3,$$

which basically reduces to the same identity as above, namely to

$$\frac{1}{9}+\frac{n^2+n+1}{(n-1)^3}=\frac{(n+2)^3}{9(n-1)^3}.$$

E27. *Let i denote the imaginary unit. Evaluate*

$$\prod_{k=1}^{n}\frac{1+i+k(k+1)}{1-i+k(k+1)}.$$

Solution 1. Factoring we see that the given product is

$$\begin{aligned}\prod_{k=1}^{n}\frac{1+i+k(k+1)}{1-i+k(k+1)}&=\prod_{k=1}^{n}\frac{(k+i)(k+1-i)}{(k+1+i)(k-i)}\\&=\prod_{k=1}^{n}\frac{k+i}{k+1+i}\cdot\prod_{k=1}^{n}\frac{k+1-i}{k-i}.\end{aligned}$$

In this splitting both products telescope and we find that the product is

$$\prod_{k=1}^{n}\frac{1+i+k(k+1)}{1-i+k(k+1)}=\frac{(1+i)(n+1-i)}{(n+1+i)(1-i)}=\frac{2(n+1)}{n^2+2n+2}+i\frac{n(n+2)}{n^2+2n+2},$$

where we have noticed that $1+i=i(1-i)$ to cancel and multiplied both numerator and denominator by $n+1-i$ to make the denominator real.

Solution 2. We can also solve the problem in a harder (but, we think, instructive) way. We have

$$1+i+k(k+1)=k^2+k+1+i=r_k(\cos t_k+i\sin t_k),$$

where

$$r_k = \sqrt{(k^2+k+1)^2+1}$$

is the modulus and

$$t_k = \arctan\frac{1}{k^2+k+1}$$

is the argument of the above complex number. Similarly,

$$1-i+k(k+1) = k^2+k+1-i = r_k(\cos t_k - i\sin t_k),$$

thus

$$\frac{1+i+k(k+1)}{1-i+k(k+1)} = \frac{r_k(\cos t_k + i\sin t_k)}{r_k(\cos t_k - i\sin t_k)} = \cos(2t_k) + i\sin(2t_k),$$

and

$$\prod_{k=1}^{n}\frac{1+i+k(k+1)}{1-i+k(k+1)} = \prod_{k=1}^{n}(\cos(2t_k)+i\sin(2t_k))$$
$$= \cos\left(2\sum_{k=1}^{n}t_k\right) + i\sin\left(2\sum_{k=1}^{n}t_k\right).$$

Now we have

$$t_k = \arctan\frac{1}{k^2+k+1} = \arctan\frac{1}{k} - \arctan\frac{1}{k+1},$$

hence

$$\sum_{k=1}^{n}t_k = \arctan 1 - \arctan\frac{1}{n+1} = \frac{\pi}{4} - \arctan\frac{1}{n+1},$$

and

$$\prod_{k=1}^{n}\frac{1+i+k(k+1)}{1-i+k(k+1)} = \sin\left(2\arctan\frac{1}{n+1}\right) + i\cos\left(2\arctan\frac{1}{n+1}\right).$$

We also used $\cos(\pi/2-t) = \sin t$ and $\sin(\pi/2-t) = \cos t$.

Explain how the results of the two solutions fit. Also, remember that we evaluated

$$\sum_{k=0}^{n} \arctan \frac{1}{k^2+k+1} = \arctan(n+1)$$

in Example 2.47, yielding

$$\sum_{k=1}^{n} \arctan \frac{1}{k^2+k+1} = \arctan(n+1) - \frac{\pi}{4}.$$

However, in this solution we get

$$\sum_{k=1}^{n} \arctan \frac{1}{k^2+k+1} = \frac{\pi}{4} - \arctan \frac{1}{n+1}.$$

Why are these two results one and the same?

E28. *Prove that the identity*

$$(x+y+z)^5-(x^5+y^5+z^5) = 5(x+y)(x+z)(y+z)(x^2+y^2+z^2+xy+xz+yz)$$

holds for any numbers x, y, *and* z.

Solution. Transforming sums into products (that is, factorizing) is always a question of interest in mathematics. But this is not the only reason for which we consider this problem in a book about sums and products. The solution that follows is in connection with the so called sums of powers. More precisely, let us denote

$$S = x+y+z, \quad Q = xy+xz+yz, \quad P = xyz$$

and

$$S_k = x^k + y^k + z^k,$$

for integer k. (S_k is thus the sum of kth powers of x, y, and z.) According to the fundamental theorem of symmetric polynomials, every such a polynomial in three variables can be expressed as a polynomial in the

fundamental symmetric polynomials, which are S, Q, and P. In particular, S_k must have such expressions whenever k is a *positive* integer. For instance $S_1 = S$,

$$S_2 = (x + y + z)^2 - 2(xy + xz + yz) = S^2 - 2Q,$$

and

$$S_3 = S^3 - 3SQ + 3P$$

(as we will see in a moment). Of course, $S_0 = 3$.

(Note that similar expressions can be found for negative integer k, too, but as rational fractions. For instance,

$$S_{-1} = Q/P \quad \text{or} \quad S_2 = (Q^2 - 2PS)/P^2$$

are immediate formulae, but this is not our matter of interest for now.)

Coming back to our problem, we see by direct computation that

$$f(t) = (t - x)(t - y)(t - z) = t^3 - St^2 + Qt - P.$$

Consequently,

$$x^3 - Sx^2 + Qx - P = f(x) = 0,$$

which implies that

$$x^k - Sx^{k-1} + Qx^{k-2} - Px^{k-3} = 0$$

for all k. Of course, similar equalities hold for y and z, too. Adding the three such equations yields

$$S_k - SS_{k-1} + QS_{k-2} - PS_{k-3} = 0 \Leftrightarrow S_k = SS_{k-1} - QS_{k-2} + PS_{k-3}$$

for all integers k, that is, we get a recurrence relation that helps us express S_k in terms of S, Q, and P for any k. Thus

$$\begin{aligned} S_3 &= SS_2 - QS_1 + PS_0 \\ &= S(S^2 - 2Q) - QS + 3P \\ &= S^3 - 3SQ + 3P, \end{aligned}$$

as we already said. We go further with S_4:

$$\begin{aligned} S_4 &= SS_3 - QS_2 + PS_1 \\ &= S(S^3 - 3SQ + 3P) - Q(S^2 - 2Q) + PS \\ &= S^4 - 4S^2Q + 4PS + 2Q^2, \end{aligned}$$

then we get

$$\begin{aligned} S_5 &= SS_4 - QS_3 + PS_2 \\ &= S(S^4 - 4S^2Q + 4PS + 2Q^2) - Q(S^3 - 3SQ + 3P) + P(S^2 - 2Q) \\ &= S^5 - 5S^3Q + 5PS^2 + 5SQ^2 - 5PQ. \end{aligned}$$

Consequently,

$$\begin{aligned} (x+y+z)^5 - (x^5 + y^5 + z^5) &= S^5 - S_5 \\ &= 5S^3Q - 5PS^2 - 5SQ^2 + 5PQ \\ &= 5(S^2 - Q)(SQ - P) \end{aligned}$$

and in this form it is not hard to factorize, is it? But

$$S^2 - Q = (x+y+z)^2 - (xy + xz + yz) = x^2 + y^2 + z^2 + xy + xz + yz$$

and

$$\begin{aligned} SQ - P &= (x+y+z)(xy+xz+yz) - xyz = \\ &= x^2y + xy^2 + x^2z + xz^2 + y^2z + yz^2 + 2xyz \\ &= (x+y)(x+z)(y+z) \end{aligned}$$

(get the factorization!) hence we obtain the required identity.

Note that, by going on in the same manner we will get

$$S^7 - S_7 = 7(SQ - P)(S^4 - 2S^2Q + PS + Q^2),$$

hence another (hard) identity:

$$\begin{aligned} (x+y+z)^7 - (x^7 + y^7 + z^7) &= 7(x+y)(x+z)(y+z) \\ &\cdot \left(\left(x^2 + y^2 + z^2 + xy + xz + yz\right)^2 + xyz(x+y+z) \right). \end{aligned}$$

If one lets $z = 0$ in these identities, then one gets other (not easy to prove and useful) identities. Find them!

E29. *Evaluate the following sum for every positive integer n*

$$1 + \cos\frac{\pi}{n} + \cos\frac{2\pi}{n} + \cdots + \cos\frac{(n-1)\pi}{n}.$$

Solution 1. For a real number x and a positive integer n, we denote

$$A = 1 + \cos x + \cos 2x + \cdots + \cos(n-1)x$$

and

$$B = \sin x + \sin 2x + \cdots + \sin(n-1)x.$$

We have

$$\begin{aligned} A + Bi &= 1 + (\cos x + i\sin x) + (\cos 2x + i\sin 2x) \\ &\quad + \cdots + (\cos(n-1)x + i\sin(n-1)x) \\ &= 1 + z + z^2 + \cdots + z^{n-1}, \end{aligned}$$

for $z = \cos x + i\sin x$ (hence $z^k = \cos kx + i\sin kx$, by de Moivre's formula). Now

$$\begin{aligned} 1 + z + z^2 + \cdots + z^{n-1} &= \frac{1-z^n}{1-z} = \frac{1 - \cos nx - i\sin nx}{1 - \cos x - i\sin x} \\ &= \frac{-2i\sin\frac{nx}{2}\left(\cos\frac{nx}{2} + i\sin\frac{nx}{2}\right)}{-2i\sin\frac{x}{2}\left(\cos\frac{x}{2} + i\sin\frac{x}{2}\right)} \\ &= \frac{\sin\frac{nx}{2}}{\sin\frac{x}{2}}\left(\cos\frac{(n-1)x}{2} + i\sin\frac{(n-1)x}{2}\right), \end{aligned}$$

because, by the usual formulas,

$$1 - \cos x - i\sin x = 2\sin^2\frac{x}{2} - 2i\sin\frac{x}{2}\cos\frac{x}{2} = -2i\sin\frac{x}{2}\left(\cos\frac{x}{2} + i\sin\frac{x}{2}\right).$$

Thus we obtained

$$A + Bi = \frac{\sin \dfrac{nx}{2}}{\sin \dfrac{x}{2}} \left(\cos \frac{(n-1)x}{2} + i \sin \frac{(n-1)x}{2} \right),$$

which, by equating the real and imaginary parts, yields

$$A = \sum_{k=0}^{n-1} \cos kx = \frac{\sin \dfrac{nx}{2} \cos \dfrac{(n-1)x}{2}}{\sin \dfrac{x}{2}}$$

and

$$B = \sum_{k=0}^{n-1} \sin kx = \frac{\sin \dfrac{nx}{2} \sin \dfrac{(n-1)x}{2}}{\sin \dfrac{x}{2}}.$$

In particular, for $x = \pi/n$, the first formula gives

$$1 + \cos \frac{\pi}{n} + \cos \frac{2\pi}{n} + \cdots + \cos \frac{(n-1)\pi}{n} = \frac{\sin \dfrac{\pi}{2} \cos \dfrac{(n-1)\pi}{2n}}{\sin \dfrac{\pi}{2n}} = 1,$$

since

$$\sin \frac{\pi}{2} = 1 \quad \text{and} \quad \cos \frac{(n-1)\pi}{2n} = \cos\left(\frac{\pi}{2} - \frac{\pi}{2n}\right) = \sin \frac{\pi}{2n}.$$

This suggests that the original problem has a simpler solution. And, indeed, we have

$$\sum_{k=1}^{n-1} \cos \frac{k\pi}{n} = \sum_{k=1}^{n-1} \cos \frac{(n-k)\pi}{n} = \sum_{k=1}^{n-1} \cos \left(\pi - \frac{k\pi}{n}\right) = -\sum_{k=1}^{n-1} \cos \frac{k\pi}{n}$$

(because when k runs from 1 to $n-1$, $n-k$ does precisely the same thing), implying our result

$$\sum_{k=1}^{n-1} \cos \frac{k\pi}{n} = 0 \Leftrightarrow \sum_{k=0}^{n-1} \cos \frac{k\pi}{n} = 1.$$

However, the calculation of the more general sums A and B is important and interesting; that's why we come with two more solutions.

Solution 2. We can evaluate (as we already have done before, but repeating is not a bad thing)

$$\begin{aligned} 2A\sin\frac{x}{2} &= \sum_{k=0}^{n-1} 2\cos kx \sin\frac{x}{2} = \sum_{k=0}^{n-1}\left(\sin\left(kx+\frac{x}{2}\right)-\sin\left(kx-\frac{x}{2}\right)\right) \\ &= \sum_{k=0}^{n-1}\left(\sin\left((k+1)x-\frac{x}{2}\right)-\sin\left(kx-\frac{x}{2}\right)\right) \\ &= \sin\left(nx-\frac{x}{2}\right)-\sin\left(-\frac{x}{2}\right) = 2\sin\frac{nx}{2}\cos\frac{(n-1)x}{2}, \end{aligned}$$

and we find again the above result. We invite the reader to proceed in the same way, in order to recapture the formula for B.

Solution 3. Once we know the formula, we can also use induction. We prove the formula for the sum B:

$$\sum_{k=0}^{n-1}\sin kx = \frac{\sin\dfrac{nx}{2}\sin\dfrac{(n-1)x}{2}}{\sin\dfrac{x}{2}}.$$

This is clear for $n=0$, or for $n=1$. Assuming it to be true for n we would get it for $n+1$ instead of n if we showed that

$$\frac{\sin\dfrac{nx}{2}\sin\dfrac{(n-1)x}{2}}{\sin\dfrac{x}{2}} + \sin nx = \frac{\sin\dfrac{(n+1)x}{2}\sin\dfrac{nx}{2}}{\sin\dfrac{x}{2}},$$

or, equivalently,

$$\sin\frac{nx}{2}\sin\frac{(n-1)x}{2} + \sin nx\sin\frac{x}{2} = \sin\frac{(n+1)x}{2}\sin\frac{nx}{2}.$$

And this is true, because

$$\sin\frac{nx}{2}\sin\frac{(n-1)x}{2} + \sin nx\sin\frac{x}{2}$$

$$= \sin\frac{nx}{2}\sin\frac{(n-1)x}{2} + 2\sin\frac{nx}{2}\cos\frac{nx}{2}\sin\frac{x}{2}$$

$$= \sin\frac{nx}{2}\left(\sin\frac{(n-1)x}{2} + 2\cos\frac{nx}{2}\sin\frac{x}{2}\right) = \sin\frac{(n+1)x}{2}\sin\frac{nx}{2},$$

where, for the last equality, we used

$$\sin(a-b) + 2\cos a\sin b = \sin(a+b)$$

for $a = nx/2$ and $b = x/2$.

Finally, observe that formulas like

$$\sum_{k=1}^{n-1}\cos kx = \frac{\sin\dfrac{(n-1)x}{2}\cos\dfrac{nx}{2}}{\sin\dfrac{x}{2}}$$

or

$$\frac{1}{2} + \sum_{k=1}^{n-1}\cos kx = \frac{\sin\dfrac{(2n-1)x}{2}}{2\sin\dfrac{x}{2}}$$

can be easily obtained from the above formula for B, by the the usual trigonometric transformations.

E30. *Evaluate*

$$\sum_{k=1}^{n}(-1)^{\frac{k(k+1)}{2}}.$$

Solution. One sees immediately that $k(k+1)/2$ is even if and only if the remainder of k when divided by 4 is either 0, or 3. Thus the sum actually can be expressed as

$$\sum_{k=1}^{n}(-1)^{\frac{k(k+1)}{2}} = -1-1+1+1-1-1+1+1-\cdots,$$

(it starts with two -1s followed by two 1s, and this pattern goes on and on), thus a quick answer would be

$$\sum_{k=1}^{n}(-1)^{\frac{k(k+1)}{2}} = \begin{cases} 0, & n \equiv 0 \bmod 4 \\ -1, & n \equiv 1 \bmod 4 \\ -2, & n \equiv 2 \bmod 4 \\ -1, & n \equiv 3 \bmod 4. \end{cases}$$

Although very rapid, this answer is also a little bit naive, as it does not offer a unitary closed form for the given sum. We will see two ways to do that.

First, if we write s_n for the given sum, we immediately note that $s_1 = -1$, $s_2 = -2$, $s_3 = -1$, $s_4 = 0$, and $s_{n+4} = s_n$ for all $n \geq 1$. This is a linear homogeneous recurrence relation, of order 4, with the characteristic equation $z^4 = 1$ having four complex (and distinct) solutions ± 1 and $\pm i$. Thus there exist constants a, b, c, and d such that

$$s_n = a + b(-1)^n + ci^n + d(-i)^n,$$

for all $n \geq 1$. The initial conditions produce the equations

$$a - b + ci - di = -1,\ a + b - c - d = -2,$$
$$a - b - ci + di = -1, \text{ and } a + b + c + d = 0,$$

which are easily solved to yield $a = -1$, $b = 0$, $c = d = 1/2$. Consequently,

$$s_n = -1 + \frac{1}{2}(i^n + (-i)^n).$$

Further

$$i^n + (-i)^n = \cos\frac{n\pi}{2} + i\sin\frac{n\pi}{2} + \cos\frac{n\pi}{2} - i\sin\frac{n\pi}{2} = 2\cos\frac{n\pi}{2},$$

thus we finally get this (maybe a little surprising) formula

$$\sum_{k=1}^{n}(-1)^{\frac{k(k+1)}{2}} = -1 + \cos\frac{n\pi}{2},$$

for every positive integer n.

Another way to proceed is to observe that

$$\sum_{k=1}^{n}(-1)^{\frac{k(k+1)}{2}} = \sum_{4j\le n} 1 + \sum_{4j-1\le n} 1 - \sum_{4j-2\le n} 1 - \sum_{4j-3\le n} 1,$$

where the first sum from the right hand side is over all indices $j \ge 1$ (positive integers) satisfying $4j \le n$ and it comprises all terms from the initial sum of the form $(-1)^{\frac{k(k+1)}{2}}$ with $k = 4j$. Similarly, the other three sums consist of the terms of s_n corresponding to indices congruent to 3 (respectively to 2, respectively to 1) modulo 4. Since there are $[n/4]$ values of $j \ge 1$ satisfying $4j \le n$, the first sum from the right equals $[n/4]$, and similarly we evaluate the other sums, getting the result

$$\sum_{k=1}^{n}(-1)^{\frac{k(k+1)}{2}} = \left\lfloor\frac{n}{4}\right\rfloor + \left\lfloor\frac{n+1}{4}\right\rfloor - \left\lfloor\frac{n+2}{4}\right\rfloor - \left\lfloor\frac{n+3}{4}\right\rfloor.$$

Again, this is somehow unexpected, isn't it? However, we obtained it in a very natural way – thus it actually is very natural, too. Also note that the nice identity

$$\left\lfloor\frac{n}{4}\right\rfloor + \left\lfloor\frac{n+1}{4}\right\rfloor - \left\lfloor\frac{n+2}{4}\right\rfloor - \left\lfloor\frac{n+3}{4}\right\rfloor = -1 + \cos\frac{n\pi}{2},\ n \in \mathbb{N}^*,$$

follows since the two results must be equal.

E31. *Evaluate*

$$\sum_{k=1}^{n}(-1)^{\frac{k(k+1)}{2}} k.$$

Solution. As in the previous problem, we immediately (and naively) find the result

$$\sum_{k=1}^{n}(-1)^{\frac{k(k+1)}{2}} k = \begin{cases} n, & n \equiv 0 \bmod 4 \\ -1, & n \equiv 1 \bmod 4 \\ -n-1, & n \equiv 2 \bmod 4 \\ 0, & n \equiv 3 \bmod 4. \end{cases}$$

But, again, this is not satisfactory enough.

We can use an Abel type summation formula (a very simple case), namely

$$\sum_{k=1}^{n} ka_k = \sum_{k=1}^{n}(a_k + \cdots + a_n) = n(a_1 + \cdots + a_n) - \sum_{k=1}^{n}(a_1 + \cdots + a_{k-1}).$$

For $a_k = (-1)^{\frac{k(k+1)}{2}}$ we obtain, according to the results of the previous problems,

$$\begin{aligned}\sum_{k=1}^{n}(-1)^{\frac{k(k+1)}{2}}k &= n\left(-1+\cos\frac{n\pi}{2}\right) - \sum_{k=1}^{n}\left(-1+\cos\frac{(k-1)\pi}{2}\right)\\ &= n\cos\frac{n\pi}{2} - \sum_{k=1}^{n}\cos\frac{(k-1)\pi}{2}\\ &= n\cos\frac{n\pi}{2} - \frac{\sin\dfrac{n\pi}{4}\cos\dfrac{(n-1)\pi}{4}}{\sin\dfrac{\pi}{4}}.\end{aligned}$$

This can also be written as

$$\sum_{k=1}^{n}(-1)^{\frac{k(k+1)}{2}}k = n\cos\frac{n\pi}{2} - \frac{\sin\dfrac{(2n-1)\pi}{4}}{2\sin\dfrac{\pi}{4}} - \frac{1}{2}$$

after using a product-to-sum formula for the numerator of the second fraction in the first form of the result.

We can also write the sum in the form

$$\sum_{k=1}^{n}(-1)^{\frac{k(k+1)}{2}}k = \sum_{4j\le n}4j + \sum_{4j-1\le n}(4j-1) - \sum_{4j-2\le n}(4j-2) - \sum_{4j-3\le n}(4j-3)$$

and the reader is invited to check that this leads to the following formula

with integral parts:

$$\sum_{k=1}^{n}(-1)^{\frac{k(k+1)}{2}}k = 2\left(\left\lfloor\frac{n}{4}\right\rfloor^2 + \left\lfloor\frac{n+1}{4}\right\rfloor^2 - \left\lfloor\frac{n+2}{4}\right\rfloor^2 - \left\lfloor\frac{n+3}{4}\right\rfloor^2\right)$$
$$+ 2\left\lfloor\frac{n}{4}\right\rfloor + \left\lfloor\frac{n+1}{4}\right\rfloor + \left\lfloor\frac{n+3}{4}\right\rfloor.$$

Observe that we can equate the two results and get another (not so nice as in the previous problem) identity. Finally, we invite the reader to see that the following equalities

$$\sum_{k=1}^{n}(-1)^k k = \left\lfloor\frac{n}{2}\right\rfloor\left(\left\lfloor\frac{n}{2}\right\rfloor + 1\right) - \left\lfloor\frac{n+1}{2}\right\rfloor^2$$
$$= (-1)^n\left\lfloor\frac{n+1}{2}\right\rfloor = \frac{(-1)^n(2n+1)-1}{4}$$

hold, by using techniques similar to the ones above. Also, observe that any of these formulae can be proved by induction – we only need to know the formula!

E32. *Prove that*

$$\sum_{k=1}^{n}\frac{(-1)^{k+1}}{1^2-2^2+3^2-\cdots+(-1)^{k+1}k^2} = \frac{2n}{n+1}.$$

Solution. For any positive integer m we have (as i^2 can be replaced by the sum of the first i odd positive integers)

$$\sum_{i=1}^{m}(-1)^{i+1}i^2 = \sum_{i=1}^{m}(-1)^{i+1}\sum_{j=1}^{i}(2j-1) = \sum_{j=1}^{m}(2j-1)\sum_{i=j}^{m}(-1)^{i+1}$$
$$= \sum_{j=1}^{m}(2j-1)\frac{(-1)^{j+1}+(-1)^{m+1}}{2}$$
$$= \sum_{j=1}^{m}(-1)^{j+1}j - \frac{1}{2}\sum_{j=1}^{m}(-1)^{j+1} + \frac{(-1)^{m+1}}{2}\sum_{j=1}^{m}(2j-1).$$

By using the already known results

$$\sum_{j=1}^{m}(-1)^j j = \frac{(-1)^m(2m+1)-1}{4}, \quad \sum_{j=1}^{m}(-1)^j = \frac{-1+(-1)^m}{2},$$

and, again, the sum of the first m odd positive integers, we get

$$\sum_{i=1}^{m}(-1)^{i+1}i^2 = (-1)^{m+1}\frac{m(m+1)}{2}.$$

Now the required sum becomes

$$\sum_{k=1}^{n}\frac{(-1)^{k+1}}{\sum_{i=1}^{k}(-1)^{i+1}i^2} = \sum_{k=1}^{n}\frac{2}{k(k+1)} = 2\sum_{k=1}^{n}\left(\frac{1}{k}-\frac{1}{k+1}\right)$$
$$= 2\left(1-\frac{1}{n+1}\right) = \frac{2n}{n+1},$$

as desired (probably the simplest and the most known telescoping sum). Note that during the proof we also used the geometric series

$$\sum_{i=s}^{t}(-1)^i = \frac{(-1)^s+(-1)^t}{2},$$

for nonnegative integers $s \le t$, a formula that, we are sure, the reader will be able to handle by her/himself.

Moreover, observe that, starting with

$$\sum_{i=1}^{m}(-1)^{i+1}i^2 = \sum_{2j-1\le m}(2j-1)^2 - \sum_{2j\le m}(2j)^2$$
$$= \frac{p(4p^2-1)}{3} - \frac{2q(q+1)(2q+1)}{3},$$

where $p = \lfloor (n+1)/2 \rfloor$ and $q = \lfloor n/2 \rfloor$, one can also come up with the formula

$$\sum_{i=1}^{m} (-1)^{i+1} i^2 = 2(-1)^{m+1} \left\lfloor \frac{m+1}{2} \right\rfloor^2 - \left\lfloor \frac{m+1}{2} \right\rfloor.$$

Although not in this problem, the formula can be useful in other situations, and it surely is interesting by itself.

E33. *If $r_1, r_2, \ldots, r_n$ and $t_1, t_2, \ldots, t_n$ are real numbers, prove that*

$$\sum_{k=1}^{n} \sum_{l=1}^{n} r_k r_l \cos(t_k - t_l) \geq 0.$$

Solution. Let $z_k = r_k(\cos t_k + i \sin t_k)$ for $1 \leq k \leq n$. We have

$$\begin{aligned} 0 &\leq |z_1 + \cdots + z_n|^2 \\ &= (r_1 \cos t_1 + \cdots + r_n \cos t_n)^2 + (r_1 \sin t_1 + \cdots + r_n \sin t_n)^2 \\ &= \left(\sum_{k=1}^{n} r_k \cos t_k \right) \left(\sum_{l=1}^{n} r_l \cos t_l \right) + \left(\sum_{k=1}^{n} r_k \sin t_k \right) \left(\sum_{l=1}^{n} r_l \sin t_l \right) \\ &= \sum_{k=1}^{n} \sum_{l=1}^{n} r_k r_l \cos t_k \cos t_l + \sum_{k=1}^{n} \sum_{l=1}^{n} r_k r_l \sin t_k \sin t_l \\ &= \sum_{k=1}^{n} \sum_{l=1}^{n} r_k r_l (\cos t_k \cos t_l + \sin t_k \sin t_l) = \sum_{k=1}^{n} \sum_{l=1}^{n} r_k r_l \cos(t_k - t_l), \end{aligned}$$

as required.

E34. *Prove that*

$$\left(\sqrt{3} + \tan 1^\circ\right) \left(\sqrt{3} + \tan 2^\circ\right) \cdots \left(\sqrt{3} + \tan 29^\circ\right) = 2^{29}.$$

Solution. We use the formula

$$\tan a + \tan b = \frac{\sin(a+b)}{\cos a \cos b}.$$

Accordingly we have

$$\sqrt{3}+\tan k^\circ=\tan 60^\circ+\tan k^\circ=\frac{\sin(60^\circ+k^\circ)}{\cos 60^\circ\cos k^\circ}=\frac{2\sin(60^\circ+k^\circ)}{\sin(90^\circ-k^\circ)}.$$

Consequently

$$\prod_{k=1}^{29}\left(\sqrt{3}+\tan k^\circ\right)=\prod_{k=1}^{29}\frac{2\sin(60^\circ+k^\circ)}{\sin(90^\circ-k^\circ)}=2^{29},$$

because the sines from the numerator are the same as those from the denominator, and they all simplify.

E35. *Evaluate*

$$(1-\cot 1^\circ)(1-\cot 2^\circ)\cdots(1-\cot 44^\circ).$$

Solution. We have

$$\begin{aligned}\cot k^\circ-1&=\frac{\cos k^\circ-\sin k^\circ}{\sin k^\circ}\\&=\frac{\sqrt{2}(\cos 45^\circ\cos k^\circ-\sin 45^\circ\sin k^\circ)}{\sin k^\circ}\\&=\frac{\sqrt{2}\cos(45^\circ+k^\circ)}{\cos(90^\circ-k^\circ)},\end{aligned}$$

hence

$$\prod_{k=1}^{44}(1-\cot k^\circ)=\prod_{k=1}^{44}(\cot k^\circ-1)=\prod_{k=1}^{44}\frac{\sqrt{2}\cos(45^\circ+k^\circ)}{\cos(90^\circ-k^\circ)}=\left(\sqrt{2}\right)^{44}=2^{22},$$

because, as in the previous problem, the cosines from the numerator cancel with those from the denominator.

E36. *Prove that*

$$\left(1-\frac{\cos 61^\circ}{\cos 1^\circ}\right)\left(1-\frac{\cos 62^\circ}{\cos 2^\circ}\right)\cdots\left(1-\frac{\cos 119^\circ}{\cos 59^\circ}\right)=1.$$

Solution. The same idea as in the two preceding problems. Based on

$$\cos k^\circ - \cos(60^\circ + k^\circ) = 2\sin 30^\circ \sin(30^\circ + k^\circ) = \sin(30^\circ + k^\circ),$$

we get

$$\prod_{k=1}^{59}\left(1 - \frac{\cos(60^\circ + k^\circ)}{\cos k^\circ}\right) = \prod_{k=1}^{59}\frac{\sin(30^\circ + k^\circ)}{\cos k^\circ} = \prod_{k=1}^{59}\frac{\sin(30^\circ + k^\circ)}{\sin(90^\circ - k^\circ)} = 1,$$

as the same sines appear in the numerator and in the denominator.

E37. *Prove that for every integer $n > 1$,*

$$\cos\frac{2\pi}{2^n - 1}\cos\frac{4\pi}{2^n - 1}\cdots\cos\frac{2^n\pi}{2^n - 1} = \frac{1}{2^n}.$$

Solution. We deal with a more general identity, namely we have

$$\cos x \cos 2x \cdots \cos 2^{n-1}x = \frac{\sin 2^n x}{2^n \sin x},$$

for any positive integer n and any real number x different from any $s\pi$, with integer s. Indeed, this is equivalent to

$$2^n \sin x \cos x \cos 2x \cdots \cos 2^{n-1}x = \sin 2^n x,$$

and follows by repeatedly applying $2\sin t\cos t = \sin 2t$ (induction can be used, to be more rigorous). We can also provide a proof by telescoping the product, as we have already seen in Example 2.31. Now we choose $x = \frac{2\pi}{2^n-1}$, and obtain

$$\cos\frac{2\pi}{2^n - 1}\cos\frac{4\pi}{2^n - 1}\cdots\cos\frac{2^n\pi}{2^n - 1} = \frac{\sin\dfrac{2^{n+1}\pi}{2^n - 1}}{2^n\sin\dfrac{2\pi}{2^n - 1}} = \frac{1}{2^n},$$

because

$$\sin\frac{2^{n+1}\pi}{2^n - 1} = \sin\left(\frac{2\pi}{2^n - 1} + 2\pi\right) = \sin\frac{2\pi}{2^n - 1}.$$

Although the general identity is true for $n \geq 1$, in the particular case $x = \frac{2\pi}{2^n-1}$ the case $n = 1$ must be handled apart, because it would lead to $x = 2\pi$, and $\sin x = 0$. (Of course, this is very simple.)

E38. *Let n be a given positive integer and let*

$$a_k = 2\cos\frac{\pi}{2^{n-k}}, \ k = 0, 1, \ldots, n-1.$$

Prove that

$$\prod_{k=0}^{n-1}(1-a_k) = \frac{(-1)^{n-1}}{1+a_0}.$$

Solution. Note that $a_0 = 0$, and $a_n = 2\cos\frac{\pi}{2^{n-n}} = -2$ can also be defined. Now we have, as desired,

$$\prod_{k=0}^{n-1}(1-a_k) = \prod_{k=0}^{n-1}\left(-\frac{1+a_{k+1}}{1+a_k}\right) = (-1)^n\frac{1+a_n}{1+a_0}$$
$$= (-1)^n\cdot(-1) = \frac{(-1)^{n-1}}{1+a_0}.$$

Indeed, the equality

$$1-a_k = -\frac{1+a_{k+1}}{1+a_k}$$

is a particular case of

$$1-2\cos x = -\frac{1+2\cos 2x}{1+2\cos x}$$

which follows immediately from $\cos 2x = 2\cos^2 x - 1$.

E39. *The sequence $\{x_n\}_{n\geq 1}$ is defined by*

$$x_1 = \frac{1}{2}, \ x_{k+1} = x_k^2 + x_k.$$

Find the greatest integer less than

$$\frac{1}{x_1+1} + \frac{1}{x_2+1} + \cdots + \frac{1}{x_{100}+1}.$$

Solution. The answer is 1. We have

$$\frac{1}{x_{k+1}}=\frac{1}{x_k(x_k+1)}=\frac{1}{x_k}-\frac{1}{x_k+1}\Leftrightarrow\frac{1}{x_k+1}=\frac{1}{x_k}-\frac{1}{x_{k+1}},$$

therefore

$$\sum_{k=1}^{100}\frac{1}{x_k+1}=\sum_{k=1}^{100}\left(\frac{1}{x_k}-\frac{1}{x_{k+1}}\right)=\frac{1}{x_1}-\frac{1}{x_{101}}=2-\frac{1}{x_{101}}.$$

All the terms of the sequence $\{x_n\}_{n\geq 1}$ are clearly positive, hence

$$\sum_{k=1}^{100}\frac{1}{x_k+1}=2-\frac{1}{x_{101}}<2$$

follows. Also, the sequence is strictly increasing and $x_3=21/16>1$, thus $x_{101}>1$, too. This shows that

$$\sum_{k=1}^{100}\frac{1}{x_k+1}=2-\frac{1}{x_{101}}>1$$

and completes the solution.

E40. *Solve the problem left unsolved in the* Introduction. *Namely, if n is any given positive integer and f is defined by*

$$f(x)=x-\left\lfloor\frac{x}{2}\right\rfloor$$

show by telescoping that

$$\sum_{k=0}^{\infty}\left\lfloor\frac{f^{[k]}(n)}{2}\right\rfloor=n-1,$$

where $f^{[k]}$ is the kth iterate of f (that is, $f^{[k]}=f\circ f\circ\cdots\circ f$ with k appearances of f; we also consider $f^{[0]}$ to be the identity function that maps x to x, for every x).

Solution. We saw in the *Introduction* that the numbers $n_k = f^{[k-1]}(n)$ eventually become equal to 1, hence the sum is actually finite (the corresponding terms in the sum are 0). Considering $x = f^{[k]}(n)$ in the very definition of f yields

$$f(f^{[k]}(n)) = f^{[k]}(n) - \left\lfloor \frac{f^{[k]}(n)}{2} \right\rfloor \Leftrightarrow \left\lfloor \frac{f^{[k]}(n)}{2} \right\rfloor = f^{[k]}(n) - f^{[k+1]}(n).$$

Thus, if $n_{s+1} = 1$ for some $s \geq 0$, then

$$\begin{aligned}
\sum_{k=0}^{\infty} \left\lfloor \frac{f^{[k]}(n)}{2} \right\rfloor &= \sum_{k=0}^{s-1} \left\lfloor \frac{f^{[k]}(n)}{2} \right\rfloor \\
&= \sum_{k=0}^{s-1} \left(f^{[k]}(n) - f^{[k+1]}(n) \right) \\
&= f^{[0]}(n) - f^{[s]}(n) = n - 1,
\end{aligned}$$

as desired.

2 Solutions to Medium Problems

M1. *For each positive integer k, let $f(k) = 4^k + 6^k + 9^k$. Prove that for all nonnegative integers m and n, $f(2^m)$ divides $f(2^n)$ whenever m is less than or equal to n.*

Solution. By repeatedly using the identity

$$(a^2 - ab + b^2)(a^2 + ab + b^2) = a^4 + a^2b^2 + b^4,$$

we get

$$a^{2^{n+1}} + a^{2^n}b^{2^n} + b^{2^{n+1}} = \left(a^{2^{m+1}} + a^{2^m}b^{2^m} + b^{2^{m+1}}\right)$$
$$\cdot \left(a^{2^{m+1}} - a^{2^m}b^{2^m} + b^{2^{m+1}}\right)\left(a^{2^{m+2}} - a^{2^{m+1}}b^{2^{m+1}} + b^{2^{m+2}}\right)\cdots$$
$$\cdot \left(a^{2^n} - a^{2^{n-1}}b^{2^{n-1}} + b^{2^n}\right)$$

whenever $m \le n$ are nonnegative integers. This shows that, if a and b are integers, and $m \le n$, then

$$a^{2^{m+1}} + a^{2^m}b^{2^m} + b^{2^{m+1}} \text{ divides } a^{2^{n+1}} + a^{2^n}b^{2^n} + b^{2^{n+1}}.$$

In particular, for $a = 2$ and $b = 3$ this means that $f(2^m)$ divides $f(2^n)$, as required.

M2. *Evaluate*

$$1^2 + 2^2 + 3^2 - 4^2 - 5^2 + 6^2 + 7^2 + 8^2 - 9^2 - 10^2 + \cdots - 2010^2,$$

where each three consecutive + signs are followed by two − signs.

Solution. We have

$$(5k-4)^2 + (5k-3)^2 + (5k-2)^2 - (5k-1)^2 - (5k)^2 = 25k^2 - 80k + 28,$$

thus our sum is

$$\sum_{k=1}^{402}((5k-4)^2 + (5k-3)^2 + (5k-2)^2 - (5k-1)^2 - (5k)^2)$$

$$= \sum_{k=1}^{402}(25k^2 - 80k + 28) = 25\sum_{k=1}^{402} k^2 - 80\sum_{k=1}^{402} k + 28\sum_{k=1}^{402} 1$$

$$= 25\frac{402 \cdot 403 \cdot 805}{6} - 80\frac{402 \cdot 403}{2} + 28 \cdot 402 = 536926141.$$

M3. *Prove that*

$$1 + 2q + 3q^2 + \cdots + nq^{n-1} = \frac{1 - nq^n}{1 - q} + \frac{q - q^n}{(1-q)^2},$$

for every $q \neq 1$.

Solution 1. Brute force is our first approach: we multiply the given sum by $(1-q)^2$ and just calculate. Thus

$$(1-q)^2(1 + 2q + 3q^2 + \cdots + nq^{n-1})$$

$$= (1-q)(1 + 2q + 3q^2 + \cdots + nq^{n-1} - q - 2q^2 - 3q^3 - \cdots - nq^n)$$

$$= (1-q)(1 + q + q^2 + \cdots + q^{n-1} - nq^n)$$

$$= (1-q)(1 - nq^n) + (1-q)(q + q^2 + \cdots + q^{n-1})$$

$$= (1-q)(1 - nq^n) + (q - q^n).$$

Now divide in this equality by $(1-q)^2$ in order to obtain the desired result. Of course, here and elsewhere, the identity

$$(1-q)(1 + q + q^2 + \cdots + q^{s-1}) = 1 - q^s$$

is taken for granted.

Solution 2. Knowing Abel's summation leads to this second approach. We use the identity

$$\sum_{k=1}^{n} ka_k = \sum_{k=1}^{n}(a_k + \cdots + a_n).$$

for $a_k = q^{k-1}$ and obtain

$$\begin{aligned}\sum_{k=1}^{n} kq^{k-1} &= \sum_{k=1}^{n} q^{k-1}(1+q+\cdots+q^{n-k}) = \sum_{k=1}^{n} q^{k-1}\frac{1-q^{n-k+1}}{1-q}\\ &= \frac{1}{1-q}\sum_{k=1}^{n} q^{k-1} - \frac{nq^n}{1-q} = \frac{1-q^n}{(1-q)^2} - \frac{nq^n}{1-q}\\ &= \frac{1-nq^n}{1-q} + \frac{q-q^n}{(1-q)^2}.\end{aligned}$$

On the other hand, if we use

$$\sum_{k=1}^{n} ka_k = n(a_1+\cdots+a_n) - \sum_{k=1}^{n}(a_1+\cdots+a_{k-1})$$

(that is, the same identity as above, but rearranged), we get

$$\begin{aligned}\sum_{k=1}^{n} kq^{k-1} &= n\frac{1-q^n}{1-q} - \sum_{k=1}^{n}\frac{1-q^{k-1}}{1-q} = n\frac{1-q^n}{1-q} - \sum_{k=2}^{n}\frac{1-q^{k-1}}{1-q}\\ &= \frac{n-nq^n}{1-q} - \frac{n-1}{1-q} + \frac{q+q^2+\cdots+q^{n-1}}{1-q}\\ &= \frac{1-nq^n}{1-q} + \frac{q-q^n}{(1-q)^2}.\end{aligned}$$

Basically, we made the same calculations in each of these approaches – they differ only formally.

Solution 3. For the reader with some basic knowledge in calculus, the following solution is available, too. We can consider the functions f and g defined, for $x \neq 1$, by

$$f(x) = 1 + x + \cdots + x^n \quad \text{and} \quad g(x) = \frac{1-x^{n+1}}{1-x}.$$

We established that $f(x) = g(x)$ for all real numbers $x \neq 1$ (actually, even in the case $x = 1$ the equality remains true if we consider instead

of $g(1)$ the limit of g for x tending to 1). Since f and g are equal, and both functions are differentiable, their derivatives must be equal: $f'(x) = g'(x)$, that is

$$\begin{aligned}1 + 2x + \cdots + nx^{n-1} &= \frac{-(n+1)x^n(1-x) + 1 - x^{n+1}}{(1-x)^2} \\ &= \frac{1-nx^n}{1-x} + \frac{x-x^n}{(1-x)^2}\end{aligned}$$

for every $x \neq 1$. We used x instead of q in this solution, since it is a more common notation for a variable.

Finally we invite the reader to take a fourth (obvious) way for solving this exercise, namely to use induction.

M4. *Evaluate*

$$\sum_{k=1}^{n} \frac{k}{k^4 + k^2 + 1}.$$

Solution. The identity

$$(a^2 - a + 1)(a^2 + a + 1) = a^4 + a^2 + 1$$

comes to our mind (and becomes handy) again. We have

$$\begin{aligned}\sum_{k=1}^{n} \frac{k}{k^4 + k^2 + 1} &= \sum_{k=1}^{n} \frac{1}{2}\left(\frac{1}{k^2 - k + 1} - \frac{1}{k^2 + k + 1}\right) \\ &= \frac{1}{2}\sum_{k=1}^{n}\left(\frac{1}{k^2 - k + 1} - \frac{1}{(k+1)^2 - (k+1) + 1}\right) \\ &= \frac{1}{2}\left(1 - \frac{1}{(n+1)^2 - (n+1) + 1}\right) = \frac{n^2 + n}{2(n^2 + n + 1)}.\end{aligned}$$

M5. *Evaluate the sum*

$$\frac{1}{3+1} + \frac{2}{3^2+1} + \frac{2^2}{3^4+1} + \cdots + \frac{2^n}{3^{2^n}+1}.$$

Solution. As simple as it may seem, it is important to notice that

$$(3^{2^k}-1)(3^{2^k}+1)=3^{2^{k+1}}-1.$$

Consequently

$$\begin{aligned}\frac{2^k}{3^{2^k}+1}&=\frac{2^k(3^{2^k}-1)}{3^{2^{k+1}}-1}=\frac{2^k(3^{2^k}+1-2)}{3^{2^{k+1}}-1}=\frac{2^k(3^{2^k}+1)-2^{k+1}}{3^{2^{k+1}}-1}\\&=\frac{2^k(3^{2^k}+1)}{3^{2^{k+1}}-1}-\frac{2^{k+1}}{3^{2^{k+1}}-1}=\frac{2^k}{3^{2^k}-1}-\frac{2^{k+1}}{3^{2^{k+1}}-1},\end{aligned}$$

hence, by telescoping,

$$\sum_{k=0}^{n}\frac{2^k}{3^{2^k}+1}=\frac{2^0}{3^{2^0}-1}-\frac{2^{n+1}}{3^{2^{n+1}}-1}=\frac{1}{2}-\frac{2^{n+1}}{3^{2^{n+1}}-1}.$$

M6. *Let $f_n=2^{2^n}+1$, $n=1,2,3,\ldots$ Prove that*

$$\frac{1}{f_1}+\frac{2}{f_2}+\cdots+\frac{2^{n-1}}{f_n}<\frac{1}{3},$$

for all positive integers n.

Solution. We have

$$\frac{2^{k-1}}{2^{2^k}+1}=\frac{2^{k-1}}{2^{2^k}-1}-\frac{2^k}{2^{2^{k+1}}-1}$$

(proceed similarly as in the previous exercise), therefore

$$\sum_{k=1}^{n}\frac{2^{k-1}}{2^{2^k}+1}=\sum_{k=1}^{n}\left(\frac{2^{k-1}}{2^{2^k}-1}-\frac{2^k}{2^{2^{k+1}}-1}\right)=\frac{1}{3}-\frac{2^n}{2^{2^{n+1}}-1}<\frac{1}{3}.$$

M7. *Let $a_n=3n+\sqrt{n^2-1}$ and $b_n=2(\sqrt{n^2-n}+\sqrt{n^2+n})$, $n\ge 1$. Prove that*

$$\sqrt{a_1-b_1}+\sqrt{a_2-b_2}+\cdots+\sqrt{a_{49}-b_{49}}=A+B\sqrt{2},$$

for some integers A and B.

Solution. Here the formula

$$a^2+b^2+c^2-2ab-2ac+2bc=(a-b-c)^2$$

comes to help us (of course, if we manage to recognize what a, b, and c are). We have

$$\begin{aligned} a_k-b_k &= 3k+\sqrt{k^2-1}-2\sqrt{k^2-k}-2\sqrt{k^2+k} \\ &= \frac{1}{2}\left(6k+2\sqrt{k^2-1}-4\sqrt{k^2-k}-4\sqrt{k^2+k}\right) \\ &= \frac{1}{2}\left(2\sqrt{k}-\sqrt{k-1}-\sqrt{k+1}\right)^2, \end{aligned}$$

thus

$$\begin{aligned} \sum_{k=1}^{49}\sqrt{a_k-b_k} &= \frac{1}{\sqrt{2}}\sum_{k=1}^{49}\left(2\sqrt{k}-\sqrt{k-1}-\sqrt{k+1}\right) \\ &= \frac{1}{\sqrt{2}}\left(1+\sqrt{49}-\sqrt{50}\right)=-5+4\sqrt{2}, \end{aligned}$$

finishing the proof.

Observe that, when we calculated $\sqrt{a_k-b_k}$, we also used the fact that

$$2\sqrt{k}-\sqrt{k-1}-\sqrt{k+1}>0.$$

One can prove this, for example, by squaring the equivalent form

$$2\sqrt{k}>\sqrt{k-1}+\sqrt{k+1},$$

or in many other ways (Jensen's inequality is another possible approach, etc.).

M8. *Let $m \le n$ be positive integers. Prove the double inequality*

$$\begin{aligned} 2\left(\sqrt{n+1}-\sqrt{m}\right) &< \frac{1}{\sqrt{m}}+\frac{1}{\sqrt{m+1}}+\cdots+\frac{1}{\sqrt{n-1}}+\frac{1}{\sqrt{n}} \\ &< 2\left(\sqrt{n}-\sqrt{m-1}\right). \end{aligned}$$

Solution. We have

$$\sqrt{k+1}-\sqrt{k}<\frac{1}{2\sqrt{k}}<\sqrt{k}-\sqrt{k-1},$$

for $k \geq 1$. Indeed

$$\sqrt{k+1}-\sqrt{k}=\frac{1}{\sqrt{k+1}+\sqrt{k}}<\frac{1}{2\sqrt{k}}$$

$$\Leftrightarrow 2\sqrt{k}<\sqrt{k+1}+\sqrt{k} \Leftrightarrow \sqrt{k}<\sqrt{k+1},$$

and the second inequality can be proved analogously. Alternately, we can use Lagrange's mean value theorem for the function $f:[k,k+1]\to\mathbb{R}$, $f(x)=\sqrt{x}$. The derivative of f is

$$f'(x)=\frac{1}{2\sqrt{x}},$$

thus we get

$$\sqrt{k+1}-\sqrt{k}=\frac{1}{2\sqrt{c}}$$

for some $c\in(k,k+1)$. Since $c>k$ we obtain

$$\sqrt{k+1}-\sqrt{k}=\frac{1}{2\sqrt{c}}<\frac{1}{2\sqrt{k}}.$$

Now all we have to do is to add the inequalities

$$2\left(\sqrt{k+1}-\sqrt{k}\right)<\frac{1}{\sqrt{k}}<2\left(\sqrt{k}-\sqrt{k-1}\right),$$

with k running from m to n, in order to obtain the required inequalities. For example, when $m=1$, we get

$$1+\frac{1}{\sqrt{2}}+\cdots+\frac{1}{\sqrt{n}}>2\left(\sqrt{n+1}-1\right),$$

or

$$x_n=\sum_{k=1}^{n}\frac{1}{\sqrt{k}}-2\sqrt{n}>2\left(\sqrt{n+1}-\sqrt{n}-1\right)>-2,$$

for all $n \geq 1$, showing that the sequence $(x_n)_{n\geq 1}$ (for x_n defined above) is bounded from below. Since

$$x_{n+1} - x_n = \frac{1}{\sqrt{n+1}} - 2\left(\sqrt{n+1} - \sqrt{n}\right) < 0$$

for all $n \geq 1$, we see that $(x_n)_{n\geq 1}$ is strictly decreasing. Being decreasing and bounded from below, $(x_n)_{n\geq 1}$ is convergent, and has a limit between -2 and $x_1 = -1$. We invite you to prove similarly that $(y_n)_{n\geq 1}$, with

$$y_n = \sum_{k=1}^{n} \frac{1}{\sqrt[3]{k}} - \frac{3}{2}\sqrt[3]{n^2}$$

is also convergent, and to find bounds for its (finite) limit.

M9. *Let*

$$a_n = 2 - \frac{1}{n^2 + \sqrt{n^4 + \frac{1}{4}}}, \quad n = 1, 2, \ldots.$$

Prove that $\sqrt{a_1} + \sqrt{a_2} + \cdots + \sqrt{a_{119}}$ *is an integer.*

Solution. We have

$$a_k = 2 - \frac{k^2 - \sqrt{k^4 + \frac{1}{4}}}{-\frac{1}{4}} = 2 + 4k^2 - 2\sqrt{2k^4 + 1}$$

$$= (2k^2 + 2k + 1) + (2k^2 - 2k + 1) - 2\sqrt{(2k^2 + 2k + 1)(2k^2 - 2k + 1)}$$

$$= \left(\sqrt{2k^2 + 2k + 1} - \sqrt{2k^2 - 2k + 1}\right)^2$$

(remember the formula for de-nesting nested radicals!). Therefore

$$\sum_{k=1}^{119} \sqrt{a_k} = \sum_{k=1}^{119} \left(\sqrt{2k^2 + 2k + 1} - \sqrt{2k^2 - 2k + 1}\right)$$

$$= \sum_{k=1}^{119} \left(\sqrt{2(k+1)^2 - 2(k+1) + 1} - \sqrt{2k^2 - 2k + 1}\right)$$

$$= \sqrt{2 \cdot 120^2 - 2 \cdot 120 + 1} - 1 = 168,$$

because

$$\begin{aligned}2 \cdot 120^2 - 2 \cdot 120 + 1 = 2 \cdot 119 \cdot 120 + 1 &= 168 \cdot 170 + 1 \\ &= (169-1)(169+1) + 1 = 169^2.\end{aligned}$$

M10. *Prove that there is no positive integer n for which*

$$\prod_{k=1}^{n} (k^4 + k^2 + 1)$$

is a perfect square.

Solution. Again,

$$k^4+k^2+1 = (k^2+k+1)(k^2-k+1) = ((k+1)^2-(k+1)+1)(k^2-k+1),$$

hence

$$\begin{aligned}P_n = \prod_{k=1}^{n}(k^4+k^2+1) &= \prod_{k=1}^{n}((k+1)^2-(k+1)+1)(k^2-k+1) \\ &= (n^2+n+1)\left(\prod_{k=2}^{n}(k^2-k+1)\right)^2.\end{aligned}$$

If we assume that, for some positive integer n the product P_n is a square, say $P_n = m^2$, with m a positive integer, then

$$n^2+n+1 = \left(\frac{m}{\prod_{k=2}^{n}(k^2-k+1)}\right)^2$$

is the square of a rational number. However, n^2+n+1 is a positive integer, so that, if it is the square of a rational number, then it actually must be the square of an integer. But this is not possible, since the inequalities

$$n^2 < n^2+n+1 < (n+1)^2$$

show that n^2+n+1 is strictly between two consecutive squares. The contradiction thus obtained finishes our proof.

M11. *Let F_n be the nth Fibonacci number. Prove that*

$$\prod_{k=0}^{n}(F_{2^k-1}+F_{2^k+1})=F_{2^{n+1}}.$$

Solution. We use a well-known Fibonacci numbers identity:

$$F_{2m}=F_{m+1}^2-F_{m-1}^2,\ m\geq 1.$$

We have

$$F_{m+1}+F_{m-1}=\frac{F_{m+1}^2-F_{m-1}^2}{F_{m+1}-F_{m-1}}=\frac{F_{2m}}{F_m},\ m\geq 1,$$

hence

$$\prod_{k=0}^{n}(F_{2^k-1}+F_{2^k+1})=\prod_{k=0}^{n}\frac{F_{2\cdot 2^k}}{F_{2^k}}=\prod_{k=0}^{n}\frac{F_{2^{k+1}}}{F_{2^k}}=\frac{F_{2^{n+1}}}{F_1}=F_{2^{n+1}}.$$

For a proof of the identity that we used, remember the formula

$$F_n=\frac{1}{\sqrt{5}}(\alpha^n-\beta^n),\ \text{with } \alpha=\frac{1+\sqrt{5}}{2} \text{ and } \beta=\frac{1-\sqrt{5}}{2}.$$

We also have $\alpha\beta=-1$, hence

$$F_{m+1}^2-F_{m-1}^2=\frac{1}{5}(\alpha^{2m+2}+\beta^{2m+2}-2(-1)^{m+1}-\alpha^{2m-2}-\beta^{2m-2}+2(-1)^{m-1})$$

$$=\frac{1}{5}\left(\alpha^{2m}\left(\alpha^2-\frac{1}{\alpha^2}\right)+\beta^{2m}\left(\beta^2-\frac{1}{\beta^2}\right)\right)=\frac{\sqrt{5}}{5}\left(\alpha^{2m}-\beta^{2m}\right)=F_{2m}$$

(observe that $\alpha^2-1/\alpha^2=\sqrt{5}$, and $\beta^2-1/\beta^2=-\sqrt{5}$).

M12. *Let x be a real number in the interval $(-1, 1)$. Evaluate*

$$\prod_{k=0}^{\infty}(1 - x^{2^k} + x^{2^{k+1}}).$$

Solution. Our old friend, the identity

$$(1 - a + a^2)(1 + a + a^2) = 1 + a^2 + a^4$$

comes to visit us again. We have

$$\prod_{k=0}^{n}(1 - x^{2^k} + x^{2^{k+1}}) = \prod_{k=0}^{n} \frac{1 + x^{2^{k+1}} + x^{2^{k+2}}}{1 + x^{2^k} + x^{2^{k+1}}} = \frac{1 + x^{2^{n+1}} + x^{2^{n+2}}}{1 + x + x^2},$$

therefore

$$\begin{aligned}\prod_{k=0}^{\infty}(1 - x^{2^k} + x^{2^{k+1}}) &= \lim_{n\to\infty} \prod_{k=0}^{n}(1 - x^{2^k} + x^{2^{k+1}}) \\ &= \lim_{n\to\infty} \frac{1 + x^{2^{n+1}} + x^{2^{n+2}}}{1 + x + x^2} \\ &= \frac{1}{1 + x + x^2},\end{aligned}$$

because, for $x \in (-1, 1)$, we have $\lim\limits_{n\to\infty} x^n = 0$.

M13. *Let F_n be the n^{th} Fibonacci number. Evaluate*

$$\sum_{k=2}^{\infty} \frac{1}{F_{k-1}F_{k+1}}.$$

Solution. Because $F_k = F_{k+1} - F_{k-1}$, we have

$$\frac{1}{F_{k-1}F_{k+1}} = \frac{F_{k+1} - F_{k-1}}{F_{k-1}F_kF_{k+1}} = \frac{1}{F_{k-1}F_k} - \frac{1}{F_kF_{k+1}},$$

and

$$\begin{aligned}\sum_{k=2}^{\infty}\frac{1}{F_{k-1}F_{k+1}} &= \sum_{k=2}^{\infty}\left(\frac{1}{F_{k-1}F_k}-\frac{1}{F_kF_{k+1}}\right)\\ &= \lim_{n\to\infty}\sum_{k=2}^{n}\left(\frac{1}{F_{k-1}F_k}-\frac{1}{F_kF_{k+1}}\right)\\ &= \lim_{n\to\infty}\left(\frac{1}{F_1F_2}-\frac{1}{F_nF_{n+1}}\right)=\frac{1}{F_1F_2}=1.\end{aligned}$$

For the Lucas numbers L_n, defined by

$$L_0=2,\ L_1=1,\ \text{and}\ L_n=L_{n-1}+L_{n-2}$$

for $n \geq 2$ compute in the same way

$$\sum_{k=2}^{\infty}\frac{1}{L_{k-1}L_{k+1}}.$$

M14. *Let n be a nonnegative integer. Prove that*

$$\sum_{k=1}^{n}k\binom{n}{k}^2=n\binom{2n-1}{n-1}.$$

Solution. Remember that

$$k\binom{n}{k}=n\binom{n-1}{k-1}$$

for $n \geq k \geq 1$, so that

$$\begin{aligned}\sum_{k=1}^{n}k\binom{n}{k}^2 &= n\sum_{k=1}^{n}\binom{n}{k}\binom{n-1}{k-1}\\ &= n\sum_{k=1}^{n}\binom{n}{n-k}\binom{n-1}{k-1}\\ &= n\binom{2n-1}{n-1}.\end{aligned}$$

The last step follows from the fact that

$$\sum_{k=1}^{n} \binom{n}{n-k}\binom{n-1}{k-1}$$

represents the coefficient of x^{n-1} in the development of the product $(1+x)^n(1+x)^{n-1}$. However, this product equals $(1+x)^{2n-1}$, therefore the coefficient of x^{n-1} in its development actually is $\binom{2n-1}{n-1}$, and the equality is thus proved.

This equality, by the way, is nothing but a special case of Vandermonde's identity: for nonnegative integers a, b, and c, we have

$$\sum_{j+k=c} \binom{a}{j}\binom{b}{k} = \binom{a+b}{c}$$

the sum being over all nonnegative integer values of j and k that sum to c. (Remember that a binomial coefficient $\binom{p}{q}$ is 0 if $p < q$.) Prove this identity and generalize it to

$$\sum_{j_1+\cdots+j_s=c} \binom{a_1}{j_1}\cdots\binom{a_s}{j_s} = \binom{a_1+\cdots+a_s}{c},$$

for nonnegative integers $a_1, \ldots, a_s$, and c.

M15. *Let n be an odd integer greater than or equal to 5. Prove that*

$$\binom{n}{1} - 5\binom{n}{2} + 5^2\binom{n}{3} - \cdots + (-1)^{n-1}5^{n-1}\binom{n}{n}$$

is not a prime number.

Solution. First of all we note another form of the given expression. Namely, we have

$$(-4)^n = (1-5)^n = 1 - 5\binom{n}{1} + 5^2\binom{n}{2} - \cdots + (-1)^n 5^n \binom{n}{n}$$

according to the binomial formula, whence

$$N = \binom{n}{1} - 5\binom{n}{2} + 5^2\binom{n}{3} - \cdots + (-1)^{n-1}5^{n-1}\binom{n}{n}$$
$$= \frac{(-4)^n - 1}{-5} = \frac{4^n + 1}{5}$$

follows. For the last equality we used the fact that n is odd. Now we have

$$4^n + 1 = 2^{2n} + 1 = \left(2^n - 2^{\frac{n+1}{2}} + 1\right)\left(2^n + 2^{\frac{n+1}{2}} + 1\right)$$

where both parentheses are (positive) integers for $n \geq 5$ odd. Because

$$N = \frac{4^n + 1}{5} = \frac{\left(2^n - 2^{\frac{n+1}{2}} + 1\right)\left(2^n + 2^{\frac{n+1}{2}} + 1\right)}{5}$$

is a natural number, one of the factors from the numerator is surely divisible by 5. So we have either

$$N = \frac{2^n - 2^{\frac{n+1}{2}} + 1}{5}\left(2^n + 2^{\frac{n+1}{2}} + 1\right)$$

or

$$N = \frac{2^n + 2^{\frac{n+1}{2}} + 1}{5}\left(2^n - 2^{\frac{n+1}{2}} + 1\right).$$

In whichever case gives integer factors, this is an expression of N as a product of two integers greater than 1, showing that N is not a prime. We only need to prove that the factors are greater than 1. Indeed, for the smallest of them we have

$$\frac{2^n - 2^{\frac{n+1}{2}} + 1}{5} = \frac{2^{\frac{n+1}{2}}(2^{\frac{n-1}{2}} - 1) + 1}{5} \geq \frac{2^3(2^2 - 1) + 1}{5} = 5 > 1$$

because $n \geq 5$. And the proof is now complete.

M16. *Prove that for any positive integer n the number*

$$a_n = \binom{2n+1}{0}2^{2n} + \binom{2n+1}{2}2^{2n-2} \cdot 3 + \cdots + \binom{2n+1}{2n}3^n$$

is the sum of two consecutive perfect squares.

Solution. By using the binomial theorem we see that

$$a_n = \frac{1}{4}\left((2+\sqrt{3})^{2n+1} + (2-\sqrt{3})^{2n+1}\right).$$

Attempting to express a_n as a sum of two consecutive squares, say

$$a_n = x^2 + (x+1)^2,$$

we find (by solving for x) that

$$x = \frac{-1 \pm \sqrt{2a_n - 1}}{2},$$

thus we expect $2a_n - 1$ to be a square. And this is, indeed, the case:

$$\begin{aligned} 2a_n - 1 &= \frac{1}{2}\left((2+\sqrt{3})^{2n+1} + (2-\sqrt{3})^{2n-1}\right) - 1 \\ &= \frac{1}{4}\left((1+\sqrt{3})^2(2+\sqrt{3})^{2n} + (1-\sqrt{3})^2(2-\sqrt{3})^{2n} - 4\right) \\ &= \left(\frac{1}{2}\left((1+\sqrt{3})(2+\sqrt{3})^n + (1-\sqrt{3})(2-\sqrt{3})^n\right)\right)^2. \end{aligned}$$

So, we have $a_n = x^2 + (x+1)^2$ for

$$x = \frac{-1 + \frac{1}{2}\left((1+\sqrt{3})(2+\sqrt{3})^n + (1-\sqrt{3})(2-\sqrt{3})^n\right)}{2}$$

(the other solution gives, basically the same expression of a_n as the sum of two consecutive squares). All that remains to prove is that

$$b_n = \frac{1}{2}\left((1+\sqrt{3})(2+\sqrt{3})^n + (1-\sqrt{3})(2-\sqrt{3})^n\right)$$

is always an odd integer (making x an integer). This is not hard to do: we have $b_0 = 1$, $b_1 = 5$, and $b_n - 4b_{n-1} + b_{n-2} = 0$ for every $n \geq 2$. The recurrence relation inductively shows that the numbers b_n are integers,

and also that b_n has the same parity as b_{n-2}, hence any b_n is odd, precisely as b_0 and b_1 are.

We obtained the recurrence relation by using the theory of second order linear and homogeneous recurrences. Namely if $\alpha \neq \beta$ and c, d are any numbers, then the sequence with general term $x_n = c\alpha^n + d\beta^n$ satisfies the recurrence relation $px_n + qx_{n-1} + rx_{n-2} = 0$, where p, q, and r are such that the equation $px^2 + qx + r = 0$ has precisely the roots α and β. This is actually also true for $\alpha = \beta$, but the converse (that is, the fact that any sequence (x_n) that satisfies the recurrence has general term $x_n = c\alpha^n + d\beta^n$ for some constants c and d) needs the condition $\alpha \neq \beta$. Try to prove these facts, then apply them to the sequence (b_n) (with $\alpha = 2 + \sqrt{3}$ and $\beta = 2 - \sqrt{3}$).

M17. *Let n be a positive integer and a be a real number, such that $\frac{a}{\pi}$ is an irrational number. Evaluate*

$$\frac{1}{\cos a - \cos 3a} + \frac{1}{\cos a - \cos 5a} + \cdots + \frac{1}{\cos a - \cos(2n+1)a}.$$

Solution. If we use a sum-to-product formula we see that the sum to evaluate actually is

$$S_n = \frac{1}{2\sin a \sin 2a} + \frac{1}{2\sin 2a \sin 3a} + \cdots + \frac{1}{2\sin na \sin(n+1)a}$$

and it seems not so easy to find a telescope for it. So, we try some small values for n and, after a few trigonometric manipulations, we obtain

$$S_2 = \frac{\sin a + \sin 3a}{2\sin a \sin 2a \sin 3a}$$

and

$$S_3 = \frac{\sin a + \sin 3a + \sin 5a}{2\sin a \sin 3a \sin 4a},$$

suggesting the formula

$$S_n = \frac{\sin a + \sin 3a + \cdots + \sin(2n-1)a}{2\sin a \sin na \sin(n+1)a}$$

for $n \geq 1$. (Notice that the same formula works for $n = 1$, too.) Now, it is natural to try induction. The inductive step would require us to prove the equality

$$\sin(n+2)a \left(\sum_{k=1}^{n} \sin(2k-1)a\right) + \sin a \sin na = \sin na \left(\sum_{j=1}^{n+1} \sin(2j-1)a\right).$$

We invite the reader to do this checking! You will probably discover that you need a closed form formula for the sum from the numerator in the expression found for S_n – and we need it anyway, otherwise we cannot say that we solved the problem. For the sum from the numerator,

$$T_n = \sin a + \sin 3a + \cdots + \sin(2n-1)a,$$

we proceed as we already did several times in such exercises, namely we multiply by $2\sin a$:

$$\begin{aligned} 2T_n \sin a &= 2\sin^2 a + 2\sin a \sin 3a + \cdots + 2\sin a \sin(2n-1)a \\ &= 1 - \cos 2a + \cos 2a - \cos 4a + \cdots + \cos(2n-2)a - \cos 2na \\ &= 1 - \cos 2na = 2\sin^2 2na, \end{aligned}$$

so that

$$T_n = \frac{\sin^2 na}{\sin a}.$$

(Or just use the identity from the end of the solution of Example 2. 25, if you proved it.) Thus the closed form that we are looking for would be

$$S_n = \frac{\sin na}{2\sin^2 a \sin(n+1)a}.$$

And now we can solve the problem, inductively or by telescoping; we choose telescoping. Based on all the above observations we calculate

$$\frac{\sin ka}{2\sin^2 a \sin(k+1)a} - \frac{\sin(k-1)a}{2\sin^2 a \sin ka} = \frac{\sin^2 ka - \sin(k-1)a \sin(k+1)a}{2\sin^2 a \sin ka \sin(k+1)a}$$

$$= \frac{\sin^2 ka - (\sin^2 ka - \sin^2 a)}{2\sin^2 a \sin ka \sin(k+1)a} = \frac{1}{2\sin ka \sin(k+1)a}$$

(we used the formula $\sin(x-y)\sin(x+y) = \sin^2 x - \sin^2 y$).
Thus we have

$$\begin{aligned} S_n &= \sum_{k=1}^{n} \frac{1}{\cos a - \cos(2k+1)a} = \sum_{k=1}^{n} \frac{1}{2\sin ka \sin(k+1)a} \\ &= \sum_{k=1}^{n} \left(\frac{\sin ka}{2\sin^2 a \sin(k+1)a} - \frac{\sin(k-1)a}{2\sin^2 a \sin ka} \right) \\ &= \frac{\sin na}{2\sin^2 a \sin(n+1)a}, \end{aligned}$$

and the claimed formula is proved.

M18. *Prove that*

$$\frac{1}{\sin 45^\circ \sin 46^\circ} + \frac{1}{\sin 47^\circ \sin 48^\circ} + \cdots + \frac{1}{\sin 133^\circ \sin 134^\circ} = \frac{1}{\sin 1^\circ}.$$

Solution. We write the equality in the equivalent form

$$\frac{\sin 1^\circ}{\sin 45^\circ \sin 46^\circ} + \frac{\sin 1^\circ}{\sin 47^\circ \sin 48^\circ} + \cdots + \frac{\sin 1^\circ}{\sin 133^\circ \sin 134^\circ} = 1,$$

and use the identity

$$\frac{\sin(b-a)}{\sin a \sin b} = \cot a - \cot b.$$

Accordingly, we have

$$\frac{\sin 1^\circ}{\sin 45^\circ \sin 46^\circ} = \cot 45^\circ - \cot 46^\circ, \quad \frac{\sin 1^\circ}{\sin 47^\circ \sin 48^\circ} = \cot 47^\circ - \cot 48^\circ,$$

and so on, until

$$\frac{\sin 1^\circ}{\sin 133^\circ \sin 134^\circ} = \cot 133^\circ - \cot 134^\circ.$$

Now add side by side all these equalities and use

$$\cot(180^\circ - x) + \cot x = 0$$

repeatedly. Thus we have

$$\cot 47^\circ + \cot 133^\circ = \cot 49^\circ + \cot 131^\circ = \cdots = \cot 89^\circ + \cot 91^\circ = 0$$

and

$$\cot 46^\circ + \cot 134^\circ = \cot 48^\circ + \cot 132^\circ = \cdots = \cot 88^\circ + \cot 92^\circ = 0.$$

Also, $\cot 90^\circ = 0$. Therefore we get

$$\frac{\sin 1^\circ}{\sin 45^\circ \sin 46^\circ} + \frac{\sin 1^\circ}{\sin 47^\circ \sin 48^\circ} + \cdots + \frac{\sin 1^\circ}{\sin 133^\circ \sin 134^\circ}$$
$$= \cot 45^\circ + \cot 47^\circ + \cdots + \cot 133^\circ - (\cot 46^\circ + \cot 48^\circ + \cdots + \cot 134^\circ)$$
$$= \cot 45^\circ = 1,$$

and we are done.

M19. *Prove that for every positive integer n and for every real number $x \neq \frac{s\pi}{2^t}$ ($t = 0, 1, 2, \ldots, n$, s an integer),*

$$\sum_{k=1}^{m} \frac{1}{\sin 2^k x} = \cot x - \cot 2^n x.$$

Solution. We have

$$\sum_{k=1}^{n} \frac{1}{\sin 2^k x} = \sum_{k=1}^{n} \left(\cot 2^{k-1} x - \cot 2^k x\right) = \cot x - \cot 2^n x,$$

by using the formula

$$\frac{1}{\sin 2t} = \cot t - \cot 2t.$$

Indeed,

$$\frac{1}{\sin 2t} = \frac{2\cos^2 t - (\cos^2 t - \sin^2 t)}{\sin 2t} = \frac{2\cos^2 t}{\sin 2t} - \frac{\cos 2t}{\sin 2t} = \cot t - \cot 2t.$$

As a final remark, note that if one relies on the similar formula

$$\tan t = \cot t - 2\cot 2t$$

then one can derive the identity

$$\sum_{k=1}^{n} 2^{k-1} \tan 2^{k-1} t = \cot t - 2^n \cot 2^n t.$$

We invite the reader to prove these results.

M20. *Show that*

$$\frac{\sin x}{\cos x} + \frac{\sin 2x}{\cos^2 x} + \cdots + \frac{\sin nx}{\cos^n x} = \cot x - \frac{\cos(n+1)x}{\sin x \cos^n x},$$

for all $x \neq s\frac{\pi}{2}$*, where* s *is an integer.*

Solution. As in the previous problem, we can guess the telescoping formula from the final result (because it is given). Namely we have

$$\frac{\sin kx}{\cos^k x} = \frac{\cos kx}{\sin x \cos^{k-1} x} - \frac{\cos(k+1)x}{\sin x \cos^k x},$$

and, consequently,

$$\sum_{k=1}^{n} \frac{\sin kx}{\cos^k x} = \sum_{k=1}^{n} \left(\frac{\cos kx}{\sin x \cos^{k-1} x} - \frac{\cos(k+1)x}{\sin x \cos^k x} \right) = \frac{\cos x}{\sin x} - \frac{\cos(n+1)x}{\sin x \cos^n x}.$$

Prove the formula that we used!

(Its proof is just based on $\cos(a+b) = \cos a \cos b - \sin a \sin b$, for $a = kx$ and $b = x$). Of course, an inductive demonstration is available, too – but it is mainly the same as the telescoping one.

M21. *For each positive integer number n prove that*

$$\cos\frac{2\pi}{2n+1}+\cos\frac{4\pi}{2n+1}+\cdots+\cos\frac{2n\pi}{2n+1}=-\frac{1}{2}.$$

Solution 1. The numbers

$$z_k=\cos\frac{2k\pi}{2n+1}+i\sin\frac{2k\pi}{2n+1},\ k=0,1,\ldots,2n$$

are the $(2n+1)$th roots of unity, that is, they are the solutions of the equation $z^{2n+1}=1$.

By Vieta's formulae, we have their sum $z_0+z_1+\cdots+z_{2n}=0$, meaning that

$$z_1+\cdots+z_{2n}=-1,$$

(because $z_0=1$). Also, it is not hard to notice that

$$z_{n+k}=z_1^{n+k}=\frac{1}{z_1^{n+1-k}}=\frac{1}{z_{n+1-k}}$$

for $k=1,2,\ldots,n$. (We used de Moivre's formula and the fact that $z_1^{2n+1}=1$.) Thus, $z_{n+1},\ldots,z_{2n}$ are the inverses of $z_n,\ldots,z_1$ respectively, and we can rewrite the above equality as

$$z_1+\cdots+z_n+\frac{1}{z_n}+\cdots+\frac{1}{z_1}=-1,$$

or

$$\frac{1}{2}\left(z_1+\frac{1}{z_1}\right)+\cdots+\frac{1}{2}\left(z_1+\frac{1}{z_1}\right)=-\frac{1}{2}.$$

Since

$$z_k+\frac{1}{z_k}=2\cos\frac{2k\pi}{2n+1}$$

for $k=1,\ldots,n$, we see that the previous equality is precisely the desired result.

Solution 2. (As the careful reader may have already observed), we can return to the identity

$$\frac{1}{2}+\cos x+\cos 2x+\cdots+\cos nx=\frac{\sin\dfrac{(2n+1)x}{2}}{2\sin\dfrac{x}{2}}$$

and replace here $x=\dfrac{2\pi}{2n+1}$. We get 0 on the right-hand side, therefore the problem is solved.

M22. *Let $\zeta\neq 1$ be a complex number with $\zeta^{23}=1$. Evaluate*

$$\sum_{k=0}^{22}\frac{1}{1+\zeta^k+\zeta^{2k}}.$$

Solution. Let

$$\alpha=\frac{-1+i\sqrt{3}}{2}\quad\text{and}\quad\beta=\frac{-1-i\sqrt{3}}{2}$$

be the two roots of unity of order 3 which are different from 1. We have

$$1+z+z^2=(\alpha-z)(\beta-z)$$

for every complex number z, hence

$$\begin{aligned}\sum_{k=0}^{22}\frac{1}{1+\zeta^k+\zeta^{2k}}&=\sum_{k=0}^{22}\frac{1}{(\alpha-\zeta^k)(\beta-\zeta^k)}\\&=\frac{1}{\beta-\alpha}\sum_{k=0}^{22}\left(\frac{1}{\alpha-\zeta^k}-\frac{1}{\beta-\zeta^k}\right).\end{aligned}$$

On the other hand, for any complex polynomial P having the roots $z_1,\ldots,z_n$ (not necessarily distinct), we have

$$\frac{P'(z)}{P(z)}=\frac{1}{z-z_1}+\cdots+\frac{1}{z-z_n},$$

for every complex number z not equal to any of $z_1, \ldots, z_n$. In our case, for the polynomial $P = z^{23} - 1$ the roots are ζ^k, with $k = 0, 1, \ldots, 22$. (Because 23 is a prime, any 23th root of unity different from 1 is a primitive root, that is, its integer powers generate all the 23th roots of unity.) Thus the above equality reads (for $P(z) = z^{23} - 1$)

$$\sum_{k=0}^{22} \frac{1}{z - \zeta^k} = \frac{P'(z)}{P(z)} = \frac{23z^{22}}{z^{23} - 1}, \quad z \neq \zeta^k, \ k = 0, 1, \ldots, 22,$$

and yields

$$\begin{aligned} \sum_{k=0}^{22} \frac{1}{1 + \zeta^k + \zeta^{2k}} &= \frac{1}{\beta - \alpha} \sum_{k=0}^{22} \left(\frac{1}{\alpha - \zeta^k} - \frac{1}{\beta - \zeta^k} \right) \\ &= \frac{1}{\beta - \alpha} \left(\frac{23\alpha^{22}}{\alpha^{23} - 1} - \frac{23\beta^{22}}{\beta^{23} - 1} \right). \end{aligned}$$

Because $\alpha^3 = \beta^3 = 1$, $\alpha + \beta = -1$, and $\alpha\beta = 1$, we can calculate

$$\begin{aligned} \frac{\alpha^{22}}{\alpha^{23} - 1} - \frac{\beta^{22}}{\beta^{23} - 1} &= \frac{\alpha}{\alpha^2 - 1} - \frac{\beta}{\beta^2 - 1} \\ &= \frac{(\beta - \alpha)(\alpha\beta + 1)}{(\alpha\beta)^2 - (\alpha + \beta)^2 + 2\alpha\beta + 1} \\ &= \frac{2}{3}(\beta - \alpha), \end{aligned}$$

and thus the final result that we are looking for is

$$\sum_{k=0}^{22} \frac{1}{1 + \zeta^k + \zeta^{2k}} = \frac{46}{3}.$$

M23. *Prove that*

$$\lfloor x \rfloor + \left\lfloor x + \frac{1}{n} \right\rfloor + \left\lfloor x + \frac{2}{n} \right\rfloor + \cdots + \left\lfloor x + \frac{n-1}{n} \right\rfloor = \lfloor nx \rfloor,$$

for all $x \in \mathbb{R}$ and any positive integer n.

We give two approaches for proving this beautiful identity of Hermite. They both need the property $\lfloor t+p \rfloor = \lfloor t \rfloor + p$ for $t \in \mathbb{R}$ and $p \in \mathbb{Z}$, and, of course, the definition of the integer part: for any real number t, the integer part of t, denoted $\lfloor t \rfloor$ is the greatest integer not greater than the number; so, we have $\lfloor t \rfloor \in \mathbb{Z}$, and $\lfloor t \rfloor \leq t < \lfloor t \rfloor + 1$.

Solution 1. We will also use in this first solution the fact that $a \leq b$ implies $\lfloor a \rfloor \leq \lfloor b \rfloor$. Thus, because

$$x \leq x + \frac{1}{n} \leq \cdots \leq x + \frac{n-1}{n} \leq x + 1,$$

we have

$$\lfloor x \rfloor \leq \left\lfloor x + \frac{1}{n} \right\rfloor \leq \cdots \leq \left\lfloor x + \frac{n-1}{n} \right\rfloor \leq \lfloor x \rfloor + 1,$$

too. Since every number $\lfloor x + i/n \rfloor$ $(0 \leq i \leq n-1)$ is an integer the above inequalities show that there must exist a number $j \in \{0, 1, \ldots, n-1\}$ such that

$$\lfloor x \rfloor = \left\lfloor x + \frac{1}{n} \right\rfloor = \cdots = \left\lfloor x + \frac{j}{n} \right\rfloor$$

and

$$\left\lfloor x + \frac{j+1}{n} \right\rfloor = \cdots = \left\lfloor x + \frac{n-1}{n} \right\rfloor = \lfloor x \rfloor + 1,$$

implying that

$$\lfloor x \rfloor + \left\lfloor x + \frac{1}{n} \right\rfloor + \left\lfloor x + \frac{2}{n} \right\rfloor + \cdots + \left\lfloor x + \frac{n-1}{n} \right\rfloor$$
$$= (j+1)\lfloor x \rfloor + (n-j-1)(\lfloor x \rfloor + 1) = n\lfloor x \rfloor + n - j - 1.$$

On the other hand, the above expressions of the integral parts are like that if and only if

$$x + \frac{j}{n} < \lfloor x \rfloor + 1 \leq x + \frac{j+1}{n},$$

which means that

$$n\lfloor x \rfloor + n - j - 1 \leq nx < n\lfloor x \rfloor + n - j.$$

But these inequalities show that

$$\lfloor nx \rfloor = n\lfloor x \rfloor + n - j - 1,$$

therefore we obtained the same value for both sides of the identity, finishing the proof.

Solution 2. Let us consider the function

$$f(x) = \lfloor x \rfloor + \left\lfloor x + \frac{1}{n} \right\rfloor + \left\lfloor x + \frac{2}{n} \right\rfloor + \cdots + \left\lfloor x + \frac{n-1}{n} \right\rfloor - \lfloor nx \rfloor .$$

Clearly, Hermite's identity is equivalent to $f(x) = 0$ for all real numbers x. We observe that

$$f\left(x + \frac{1}{n}\right) = \left\lfloor x + \frac{1}{n} \right\rfloor + \left\lfloor x + \frac{2}{n} \right\rfloor + \cdots + \left\lfloor x + \frac{n-1}{n} \right\rfloor + \lfloor x + 1 \rfloor - \lfloor nx + 1 \rfloor = f(x)$$

because

$$\lfloor x + 1 \rfloor = \lfloor x \rfloor + 1 \quad \text{and} \quad \lfloor nx + 1 \rfloor = \lfloor nx \rfloor + 1.$$

That is, f is periodic with $1/n$ as a period. Thus it suffices to prove $f(x) = 0$ (and the identity) for $x \in [0, 1/n)$. But for such x all the integral parts involved in the definition of f are 0 (all $x + i/n$ for $1 \le i \le n-1$ and also nx are between 0 and 1) hence $f(x) = 0$ obviously follows – and it follows for all $x \in \mathbb{R}$, finishing the proof.

M24. *Prove that for every positive integer n*

$$\left\lfloor \frac{n + 2^0}{2^1} \right\rfloor + \left\lfloor \frac{n + 2^1}{2^2} \right\rfloor + \left\lfloor \frac{n + 2^2}{2^3} \right\rfloor + \cdots = n.$$

Solution. The sum is not an infinite sum, because, at some moment the integral parts become 0; more precisely, we have

$$\frac{n + 2^k}{2^{k+1}} < 1 \Leftrightarrow n < 2^k$$

hence the terms are 0 for $k > \log_2 n$. We use Hermite's identity in the form

$$\left\lfloor x + \frac{1}{2} \right\rfloor = \lfloor 2x \rfloor - \lfloor x \rfloor.$$

Accordingly,

$$\sum_{k\geq 0} \left\lfloor \frac{n+2^k}{2^{k+1}} \right\rfloor = \sum_{k\geq 0} \left\lfloor \frac{n}{2^{k+1}} + \frac{1}{2} \right\rfloor = \sum_{k\geq 0} \left(\left\lfloor \frac{n}{2^k} \right\rfloor - \left\lfloor \frac{n}{2^{k+1}} \right\rfloor \right) = \lfloor n \rfloor = n$$

(Hermite's identity allows telescoping for this sum).

M25. *Evaluate*

$$\sum_{0\leq i<j\leq n} \left\lfloor \frac{x+i}{j} \right\rfloor,$$

where x is a real number.

Solution. We have

$$\sum_{0\leq i<j\leq n} \left\lfloor \frac{x+i}{j} \right\rfloor = \sum_{j=1}^{n} \sum_{i=0}^{j-1} \left\lfloor \frac{x}{j} + \frac{i}{j} \right\rfloor = \sum_{j=1}^{n} \left\lfloor j \frac{x}{j} \right\rfloor = n\lfloor x \rfloor,$$

where, for each inner sum we used Hermite's identity.

M26. *Let x, y, and z be integers such that $xy + xz + yz = 0$. Prove that $(x+y+z)^2$ divides $x^5 + y^5 + z^5$.*

Solution. We have the identity

$$(x+y+z)^5-(x^5+y^5+z^5) = 5(x+y)(x+z)(y+z)(x^2+y^2+z^2+xy+xz+yz)$$

proved in problem E28 in the previous section. By $xy + xz + yz = 0$ we get

$$(x+y)(x+z)(y+z) = -xyz$$

and

$$x^2 + y^2 + z^2 + xy + xz + yz = (x+y+z)^2,$$

therefore the identity becomes

$$(x+y+z)^5-(x^5+y^5+z^5)=-5xyz(x+y+z)^2.$$

Consequently

$$x^5+y^5+z^5=(x+y+z)^5+5xyz(x+y+z)^2$$

is divisible by $(x+y+z)^2$, as required.

Note that such (nontrivial) integers do exist. For instance, we can consider $x=12$, $y=4$, and $z=-3$ for which $xy+xz+yz=0$ and

$$x^5+y^5+z^5=249613=13^2\cdot 1477$$

is divisible by $13^2=(x+y+z)^2$.

Actually there are infinitely many triples of integers with the given property. For example, take $x=a(a+b)$, $y=b(a+b)$ and $z=-ab$ (with integer a and b), and we have the desired equation $xy+xz+yz=0$ fulfilled. With some little effort you can find all such triples of integers.

M27. *Let p be an odd prime. Prove that*

$$\sum_{k=1}^{p-1}\frac{k^p-k}{p}\equiv\frac{p+1}{2}\pmod{p}.$$

Solution. Note that, by Fermat's little theorem, each $(k^p-k)/p$ is a natural number. We have, by the binomial formula,

$$(k-p)^p=k^p-pk^{p-1}p+\frac{p(p-1)}{2}k^{p-2}p^2-\cdots\equiv k^p\pmod{p^2}$$

(all missing terms are clearly divisible by p^2), hence

$$k^p+(p-k)^p\equiv 0\pmod{p^2}$$

for every $k \in \{0, 1, \ldots, p-1\}$ (and for every other integer k, but we are only interested in these values). Thus

$$\sum_{k=1}^{p-1} k^p = \frac{1}{2}\left(\sum_{k=1}^{p-1} k^p + \sum_{k=1}^{p-1} k^p\right) = \frac{1}{2}\left(\sum_{k=1}^{p-1} k^p + \sum_{k=1}^{p-1}(p-k)^p\right)$$
$$= \frac{1}{2}\sum_{k=1}^{p-1}(k^p + (p-k)^p) \equiv 0 \pmod{p^2}$$

(for odd p dividing by 2 doesn't influence the congruence).

Consequently,

$$\sum_{k=1}^{p-1}(k^p - k) = \sum_{k=1}^{p-1} k^p - \sum_{k=1}^{p-1} k \equiv -\frac{p(p-1)}{2} \pmod{p^2},$$

therefore

$$\sum_{k=1}^{p-1} \frac{k^p - k}{p} \equiv -\frac{p-1}{2} \pmod{p}$$

and the conclusion follows because, evidently,

$$-\frac{p-1}{2} \equiv \frac{p+1}{2} \pmod{p}.$$

M28. *Prove that for each positive integer $n \geq 2$ the following inequality holds*

$$\sigma(n)\phi(n) < n^2,$$

where $\phi(n)$ is the number of integers that are less than n and are relatively prime with n, and $\sigma(n)$ is the sum of the positive divisors of n.

Solution. The arithmetic functions σ and ϕ are multiplicative, that is

$$\sigma(xy) = \sigma(x)\sigma(y) \quad \text{and} \quad \phi(xy) = \phi(x)\phi(y)$$

for any relatively prime positive integers x and y.

Therefore, if $n = p_1^{a_1} \cdots p_s^{a_s}$ is the factorization of n (with primes $p_1, \dots, p_s$ and positive integers $a_1, \dots, a_s$), we have

$$\sigma(n)\phi(n) = \sigma(p_1^{a_1})\phi(p_1^{a_1}) \cdots \sigma(p_s^{a_s})\phi(p_s^{a_s}) < (p_1^{a_1})^2 \cdots (p_s^{a_s})^2 = n^2,$$

provided we proved the inequality for n of the form p^a, with prime p and a a positive integer. This is not hard, namely we have

$$\begin{aligned}\sigma(p^a)\phi(p^a) &= (1 + p + p^2 + \cdots + p^a)p^{a-1}(p-1)\\ &= (p^{a+1} - 1)p^{a-1} < p^{a+1}p^{a-1} = (p^a)^2,\end{aligned}$$

and thus our proof is complete.

M29. *Let m and n be positive integers with m even and at least equal to 4. Prove that*

$$\sum_{k=0}^{m} (-4)^k n^{4(m-k)}$$

is not a prime number.

Solution. We have

$$\sum_{k=0}^{m} (-4)^k n^{4(m-k)} = \frac{n^{4(m+1)} + 4^{m+1}}{n^4 + 4}$$

by using the formula

$$\sum_{k=0}^{m} a^k b^{m-k} = \frac{a^{m+1} - b^{m+1}}{a - b} \quad (a \neq b)$$

and the fact that m is even. Now, another formula, namely

$$a^4 + 4b^4 = (a^2 - 2ab + 2b^2)(a^2 + 2ab + 2b^2)$$

shows that the numerator of the final expression of our sum factors as

$$\begin{aligned}n^{4(m+1)} + 4^{m+1} &= (n^{m+1})^4 + 4(2^{m/2})^4\\ &= (n^{2m+2} - 2^{m/2+1}n^{m+1} + 2^{m+1})(n^{2m+2} + 2^{m/2+1}n^{m+1} + 2^{m+1}).\end{aligned}$$

If we prove that both these factors are greater than the denominator, we are done (since if pq/r is an integer for positive integers p, q, and r, and $p > r$, $q > r$, then after cancellations pq/r surely remains a product of two integers greater than 1). Of course, it suffices to show that

$$n^{2m+2} - 2^{m/2+1}n^{m+1} + 2^{m+1} > n^4 + 4.$$

This follows, for $n \geq 2$, by adding

$$n^{2m+2} - 2^{m/2+1}n^{m+1} = n^{m+1}(n^{m+1} - 2^{m/2+1}) > n^{m+1} > n^4$$

with $2^{m+1} > 4$, while for $n = 1$ it reduces to $2^m - 2^{m/2} > 2$, hence to $2^{m/2} > 2$.

Note that for $m = 2$ the above inequality becomes

$$n^6 - n^4 - 4n^3 + 4 > 0 \Leftrightarrow (n-1)(n^5 + n^4 - 4n^2 - 4n - 4) > 0$$

and it still holds for $n > 1$. So, the given sum is not a prime for $m \geq 2$ even and any positive integer n with the only exception of $m = 2$ and $n = 1$ when the sum equals 13.

M30. *Let p be a prime such that $p \equiv 1 \pmod 3$ and let $q = \lfloor 2p/3 \rfloor$. If*

$$\frac{1}{1\cdot 2} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{(q-1)q} = \frac{m}{n}$$

for some integers m and n, prove that $p \mid m$.

Solution. Suppose $p = 3s + 1$, with s a positive integer. Then

$$q = \lfloor 2p/3 \rfloor = \lfloor 2s + 2/3 \rfloor = 2s,$$

and the sum from the statement of the problem actually is

$$\begin{aligned} \frac{1}{1\cdot 2} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{(2s-1)2s} &= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{2s-1} - \frac{1}{2s} \\ &= \frac{1}{s+1} + \frac{1}{s+2} + \cdots + \frac{1}{2s}. \end{aligned}$$

So, we have to prove that if we write the sum $\sum_{k=s+1}^{2s} \frac{1}{k}$ as an ordinary fraction, then the numerator of this fraction is divisible by p. But we have

$$\begin{aligned}\sum_{k=s+1}^{2s} \frac{1}{k} &= \frac{1}{2}\left(\sum_{k=s+1}^{2s} \frac{1}{k} + \sum_{k=s+1}^{2s} \frac{1}{k}\right) = \frac{1}{2}\left(\sum_{k=s+1}^{2s} \frac{1}{k} + \sum_{k=s+1}^{2s} \frac{1}{3s+1-k}\right)\\
&= \frac{1}{2}\sum_{k=s+1}^{2s}\left(\frac{1}{k} + \frac{1}{3s+1-k}\right) = \frac{1}{2}\sum_{k=s+1}^{2s} \frac{3s+1}{k(3s+1-k)}\\
&= \frac{1}{2}\sum_{k=s+1}^{2s} \frac{p}{k(3s+1-k)}.\end{aligned}$$

Thus we have written the sum as another sum in which the numerator of each term is p, and the factors from the denominators are strictly less than p (including the factor 2). This means that none of these fractions can be simplified by p, therefore the factor p remains after addition is performed, and cannot be simplified, hence the final numerator is divisible by p, as we intended to prove.

For example, when $p = 13$ we have $q = 8$ $(s = 4)$, and the sum from the statement of the problem is

$$\begin{aligned}\frac{1}{1\cdot 2} + \frac{1}{3\cdot 4} + \frac{1}{5\cdot 6} + \frac{1}{7\cdot 8} &= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8}\\
&= \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\\
&= \frac{1}{2}\left(\frac{13}{5\cdot 8} + \frac{13}{6\cdot 7} + \frac{13}{7\cdot 6} + \frac{13}{8\cdot 5}\right).\end{aligned}$$

For the sake of completeness we prove the middle equality (but we proved it before in Example 4.6 – and we suppose that the careful reader already did it again), that is, in the general case,

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{2s-1} - \frac{1}{2s} = \frac{1}{s+1} + \frac{1}{s+2} + \cdots + \frac{1}{2s}.$$

Indeed we have

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots\frac{1}{2s-1}-\frac{1}{2s}=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{2s-1}+\frac{1}{2s}$$

$$-2\left(1+\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2s}\right)=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{2s-1}+\frac{1}{2s}$$

$$-\left(1+\frac{1}{2}+\cdots+\frac{1}{s}\right)=\frac{1}{s+1}+\frac{1}{s+2}+\cdots+\frac{1}{2s}.$$

M31. *Prove that for different choices of the signs + and − the expression*

$$\pm 1 \pm 2 \pm \cdots \pm (4n+1)$$

yields all odd positive integers less than or equal to $(2n+1)(4n+1)$.

Solution. This is clearly true for $n=0$, so we will assume it true for $n-1$ and prove that it also holds for n. Thus, the induction hypothesis says that the sums

$$\pm 1 \pm 2 \pm \cdots \pm (4n-3)$$

produce (for various choices of the $\pm$ signs) all the odd positive integers at most equal to $(2n-1)(4n-3)$. Therefore, these numbers can be also achieved as sums of the form

$$\pm 1 \pm 2 \pm \cdots \pm (4n-3)+(4n-2)-(4n-1)-4n+(4n+1)$$

as required.
In what concerns the other odd positive integer until $(2n+1)(4n+1)$, we first note that

$$1+2+\cdots+(4n+1)=(2n+1)(4n+1).$$

We can subtract $2j$ (with $1 \le j \le 4n+1$) from this sum and get a representation of $(2n+1)(4n+1)-2j$, namely

$$(2n+1)(4n+1)-2j=1+2+\cdots+(j-1)-j+(j+1)+\cdots+(4n+1).$$

We have thus represented all the odd numbers (in decreasing order) from $(2n+1)(4n+1)$ until

$$(2n+1)(4n+1) - 2(4n+1) = (2n-1)(4n+1).$$

Now, by the same idea, we subtract from

$$1+2+\cdots+4n-(4n+1) = (2n-1)(4n+1)$$

all the numbers $2k$ with $1 \le k \le 4n$, and get representations for all the odd numbers

$$(2n-1)(4n+1) - 2k, \; k = 1, 2, \ldots, 4n.$$

Thus we have representations for all the odd positive integers from

$$(2n+1)(4n+1) \quad \text{until} \quad (2n-1)(4n+1) - 8n = 8n^2 - 10n - 1.$$

On the other hand, as we have seen in the beginning, all odd positive integers from 1 to $(2n-1)(4n-3) = 8n^2 - 10n + 3$ also do have representations. As $8n^2 - 10n + 3 > 8n^2 - 10n - 1$ the problem is solved.

M32. *Let n be a positive integer. Prove that all binomial coefficients $\binom{n}{k}$ with $0 \le k \le n$ are odd if and only if $n = 2^m - 1$ for some nonnegative integer m.*

Solution 1. By Lucas's theorem, a binomial coefficient $\binom{n}{k}$ is divisible by a prime p if and only if there exists $0 \le i \le s$ such that $n_i < k_i$, where $n = n_0 + n_1 p + \cdots + n_s p^s$ and $k = k_0 + k_1 p + \cdots + k_s p^s$ are the base p representations of n and k, respectively. It follows that precisely

$$(n_0+1)(n_1+1)\cdots(n_s+1)$$

of these coefficients (namely those for which $k_i \le n_i$ for every $0 \le i \le s$) are not divisible by p (and, consequently.

$$n+1-(n_0+1)(n_1+1)\cdots(n_s+1)$$

are divisible by p).

Thus the given condition that all binomial coefficients $\binom{n}{k}$ are odd can be restated as saying that

$$(n_0+1)(n_1+1)\cdots(n_s+1)=n+1,$$

where $n=n_0+n_1 2+\cdots+n_s 2^s$ is the representation of n in base 2, with $n_i\in\{0,1\}$ for all $0\le i\le s$. This is clearly true if

$$n=2^m-1=1+2+\cdots+2^{m-1}$$

has all its base 2 digits equal to 1.

Conversely, assume that the equality

$$(n_0+1)(n_1+1)\cdots(n_s+1)=n+1,$$

holds for $n=n_0+n_1 2+\cdots+n_s 2^s$. Since each n_i is either 0 or 1, this clearly gives $n+1=2^m$ with m being the number of those n_i that are equal to 1; that is, we get $n=2^m-1$, as desired.

Note that, if $n_s\ne 0$ we have $2^s\le n<2^{s+1}$, or $2^s-1<2^m-1\le 2^{s+1}-1$, hence we must actually have $m=s+1$.

For primes p other than 2, a similar argument works. If $2\le a\le p$, then

$$n=ap^m-1=(p-1)+(p-1)p+\cdots+(p-1)p^{m-1}+(a-1)p^m,$$

the equality

$$(n_0+1)(n_1+1)\cdots(n_{m-1}+1)(n_m+1)=n+1$$

holds. Conversely, these are the only cases where equality holds.

Solution 2. Actually, a shorter solution can be given. It is clear, by Lucas's theorem, that for $n=2^m-1$ all binomial coefficients are odd. On the other hand, if n is not of this form, than its base 2 representation $n=n_0+n_1 2+\cdots+n_s 2^s$ must contain a zero digit n_i with $0\le i<s$. Again by Lucas's theorem, we see that the binomial coefficient $\binom{n}{2^i}$ is even. Nevertheless, the first solution provides more information about the binomial coefficients modulo 2 (or modulo a prime p, in general).

M33. *For each positive integer n define*

$$a_n = \frac{(n+1)(n+2)\cdots(n+2010)}{2010!}.$$

Prove that there are infinitely many n such that a_n is an integer with no prime factors less than 2010.

Solution 1. Note that a_n is an integer because it is a binomial coefficient:

$$a_n = \binom{n+2010}{2010}.$$

The numbers $n_m = (2010!)^2 m$, with positive integer m will do the job. Indeed, we have

$$(n_m+1)(n_m+2)\cdots(n_m+2010) \equiv 2010! \pmod{(2010!)^2}$$

yielding that

$$\frac{(n_m+1)(n_m+2)\cdots(n_m+2010)}{2010!} \equiv 1 \pmod{2010!}.$$

In particular, this congruence holds for any prime $p < 2010$, showing that the number

$$\frac{(n_m+1)(n_m+2)\cdots(n_m+2010)}{2010!}$$

cannot be divisible by p (as being of the form $kp+1$). In place of the numbers n_m we can also take numbers of the form $P^2 m$, where P is the product of the primes less than 2010.

Solution 2. We will use again the theorem of Lucas. We consider the product P of all the primes less than 2010, and numbers of the form $n_k = P^k$, with $k \geq 11$. For any prime p less than 2010 (therefore a factor of P) the base p representation of 2010 has at most 11 digits, say

$$2010_p = d_{10}d_9 \ldots d_1 d_0.$$

On the other hand, because n_k is divisible by p^{11}, it has all its last (at least) 11 digits in base p equal to 0, say

$$(n_k)_p = d_s \dots d_{11} \underbrace{0 \dots 0}_{11 \text{ digits}}.$$

Thus,

$$(n_k + 2010)_p = d_s \dots d_{11} d_{10} \dots d_0,$$

and by Lucas's theorem

$$\binom{n_k + 2010}{2010} \equiv \binom{d_s}{0} \cdots \binom{d_{11}}{0} \binom{d_{10}}{d_{10}} \cdots \binom{d_0}{d_0} = 1 \pmod{p},$$

and we finish the proof as in the first solution.

M34. *The numbers* $a_1, a_2, \dots, a_n > 0$ *and* $b_1 \geq b_2 \geq \cdots \geq b_n > 0$ *satisfy*

$$a_1 \geq b_1, \ a_1 + a_2 \geq b_1 + b_2, \dots,$$

$$a_1 + a_2 + \cdots + a_n \geq b_1 + b_2 + \cdots + b_n.$$

Prove that for every positive integer j,

$$a_1^j + a_2^j + \cdots + a_n^j \geq b_1^j + b_2^j + \cdots + b_n^j.$$

Solution 1. We use induction on j to show that

$$a_1^j + a_2^j + \cdots + a_n^j \geq b_1^j + b_2^j + \cdots + b_n^j$$

for each positive integer j. This is clear for $j = 1$, so we assume it to be true for some j, and prove it for $j + 1$. Note that, for any $1 \leq k \leq n$ the numbers $a_1, a_2, \dots, a_k$ satisfy the same conditions as $a_1, a_2, \dots, a_n$, therefore the inductive assumption applies to them, too. Consequently we also now that

$$a_1^j + a_2^j + \cdots + a_k^j \geq b_1^j + b_2^j + \cdots + b_k^j$$

for every $k \in \{1, 2, \ldots, n\}$. Now, by Abel's summation formula (twice) and the hypotheses (including the induction hypothesis), we have

$$a_1^j b_1 + a_2^j b_2 + \cdots + a_n^j b_n$$

$$= (b_1 - b_2)a_1^j + (b_2 - b_3)(a_1^j + a_2^j) + \cdots + (b_{n-1} - b_n)(a_1^j + a_2^j + \cdots + a_{n-1}^j)$$
$$+ b_n(a_1^j + a_2^j + \cdots + a_n^j)$$
$$\geq (b_1 - b_2)b_1^j + (b_2 - b_3)(b_1^j + b_2^j) + \cdots + (b_{n-1} - b_n)(b_1^j + b_2^j + \cdots + b_{n-1}^j)$$
$$+ b_n(b_1^j + b_2^j + \cdots + b_n^j) = b_1^{j+1} + b_2^{j+1} + \cdots + b_n^{j+1}.$$

We also use the inequality between the arithmetic and geometric means to infer

$$j a_k^{j+1} + b_k^{j+1} \geq (j+1) a_k^j b_k$$

for every $1 \leq k \leq n$. Adding all these inequalities and using the above yield

$$j \sum_{k=1}^n a_k^{j+1} + \sum_{k=1}^n b_k^{j+1} \geq (j+1) \sum_{k=1}^n a_k^j b_k \geq (j+1) \sum_{k=1}^n b_k^{j+1}$$

thus

$$\sum_{k=1}^n a_k^{j+1} \geq \sum_{k=1}^n b_k^{j+1}$$

follows, finishing the proof.

Solution 2. Note that we may assume without loss of generality that $a_1 \geq a_2 \geq \cdots \geq a_n > 0$.

Consider the numbers

$$x_k = a_k - b_k \quad \text{and} \quad y_k = a_k^{j-1} + a_k^{j-2} b_k + \cdots + a_k b_k^{j-2} + b_k^{j-1}$$

for $k = 1, 2, \ldots, n$. By the hypotheses we have

$$x_1 + x_2 + \cdots + x_k \geq 0$$

for all $1 \le k \le n$ and also

$$y_1 \ge y_2 \ge \cdots \ge y_n > 0.$$

Also, $a_k^j - b_k^j = x_k y_k$ holds for every $1 \le k \le n$. Thus, by using Abel's summation formula and all the above observations, we have

$$\begin{aligned}\sum_{k=1}^{n} a_k^j - \sum_{k=1}^{n} b_k^j = \sum_{k=1}^{n} x_k y_k &= x_1(y_1 - y_2) + (x_1 + x_2)(y_1 - y_2) \\ &+ \cdots + (x_1 + x_2 + \cdots + x_{n-1})(y_{n-1} - y_n) \\ &+ (x_1 + x_2 + \cdots + x_n)y_n \ge 0\end{aligned}$$

and the conclusion follows for every $j \ge 1$.

M35. *Evaluate*

$$\sum_{k=1}^{\infty}\sum_{l=k}^{\infty}\frac{1}{l!}.$$

Solution. Remember that, for any $x \in \mathbb{R}$, we have

$$\sum_{m=0}^{\infty}\frac{x^m}{m!} = e^x,$$

the well-known formula that defines the exponential, in which the absolute convergence of the series is also well-known. Thus, in particular,

$$\sum_{m=0}^{\infty}\frac{1}{m!} = e,$$

where the series is absolutely convergent. Consequently, for the series from our problem the (absolute) convergence is clear, too, and reversing the order of summation is allowed. And we have

$$\begin{aligned}\sum_{k=1}^{\infty}\sum_{l=k}^{\infty}\frac{1}{l!} &= \sum_{k=1}^{\infty} 1 \sum_{l\ge k}\frac{1}{l!} = \sum_{l=1}^{\infty}\frac{1}{l!}\sum_{1\le k\le l} 1 = \sum_{l=1}^{\infty}\frac{1}{l!}l \\ &= \sum_{l=1}^{\infty}\frac{1}{(l-1)!} = \sum_{m=0}^{\infty}\frac{1}{m!} = e.\end{aligned}$$

(Basically, the above line says that in the double sum every term $1/l!$ appears precisely l times.)

M36. *Evaluate*

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{i!j!}{(i+j+1)!}.$$

Solution. First note that we have

$$\sum_{t=1}^{\infty}\frac{s!t!}{(s+t)!}=\frac{1}{s-1},$$

for any positive integer $s \geq 2$. Indeed,

$$\sum_{t=1}^{\infty}\frac{s!t!}{(s+t)!}=s!\sum_{t=1}^{\infty}\frac{1}{(t+1)(t+2)\cdots(t+s)}$$

$$=\frac{s!}{s-1}\sum_{t=1}^{\infty}\left(\frac{1}{(t+1)(t+2)\cdots(t+s-1)}-\frac{1}{(t+2)(t+3)\cdots(t+s)}\right)$$

$$=\frac{s!}{s-1}\frac{1}{s!}=\frac{1}{s-1}.$$

Now, for the problem, we have

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{i!j!}{(i+j+1)!}=\sum_{i=1}^{\infty}\frac{1}{i+1}\sum_{j=1}^{\infty}\frac{(i+1)!j!}{(i+1+j)!}$$

$$=\sum_{i=1}^{\infty}\frac{1}{i+1}\frac{1}{i}$$

$$=\sum_{i=1}^{\infty}\left(\frac{1}{i}-\frac{1}{i+1}\right)=1$$

according to the above result, and to the well-known result about the most ubiquitous telescopic sum, namely

$$\sum_{i=1}^{\infty}\frac{1}{i(i+1)}=1.$$

Note that the formula proved in the beginning can also be expressed as

$$\sum_{t=1}^{\infty} \frac{1}{\binom{s+t}{s}} = \frac{1}{s-1}$$

and that the sum can also be telescoped in the form

$$\begin{aligned}\sum_{t=1}^{\infty} \frac{1}{\binom{s+t}{s}} &= \frac{s}{s-1} \sum_{t=1}^{\infty} \left(\frac{1}{\binom{s+t-1}{s-1}} - \frac{1}{\binom{s+t}{s-1}} \right) \\ &= \frac{s}{s-1} \frac{1}{\binom{s}{s-1}} = \frac{1}{s-1}.\end{aligned}$$

M37. *Prove the inequality*

$$\sum_{k=1}^{\infty} \frac{1}{(k+1)\sqrt{k}} < 2.$$

We met this before as Example 1.22 (although there the inequality was stated for the partial sum of the series, basically we have the same problem) and now we come with two more solutions.

Solution 1. We have

$$\begin{aligned}\sum_{k=4}^{\infty} \frac{1}{(k+1)\sqrt{k}} &= \sum_{j=3}^{\infty} \frac{1}{(j+2)\sqrt{j+1}} < \sum_{j=3}^{\infty} \int_j^{j+1} \frac{1}{(x+1)\sqrt{x}} dx \\ &= \int_3^{\infty} \frac{1}{(x+1)\sqrt{x}} dx = 2 \arctan \sqrt{x} \Big|_3^{\infty} = \frac{\pi}{3},\end{aligned}$$

therefore

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{1}{(k+1)\sqrt{k}} &= \frac{1}{2} + \frac{1}{3\sqrt{2}} + \frac{1}{4\sqrt{3}} + \sum_{k=4}^{\infty} \frac{1}{(k+1)\sqrt{k}} \\ &< \frac{1}{2} + \frac{1}{3\sqrt{2}} + \frac{1}{4\sqrt{3}} + \frac{\pi}{3} \\ &< 0.5 + 0.4 + 1.1 = 2,\end{aligned}$$

as required, because

$$\frac{1}{3\sqrt{2}} + \frac{1}{4\sqrt{3}} < \frac{1}{3 \cdot 1.4} + \frac{1}{4 \cdot 1.7} = \frac{11}{4.2 \cdot 6.8} < 0.4 \quad (\Leftrightarrow 11 < 11.424)$$

and $\pi/3 < 1.1$.

It is not unusual to compare $\sum_{k=1}^{\infty} f(k)$ to $\int_1^{\infty} f(x)dx$ (actually, it is a standard procedure). For example, if the function $f : (0, \infty) \to \mathbb{R}$ is decreasing (as is $f(x) = 1/((x+1)\sqrt{x})$ in our example), by the mean-value theorem for the Riemann integral (or just by the monotonicity of the integral) we have

$$f(j+1) < \int_j^{j+1} f(x)dx < f(j), \ j \geq 1,$$

thus

$$\int_n^{\infty} f(x)dx < \sum_{j=n}^{\infty} f(j) < \int_{n-1}^{\infty} f(x)dx$$

follows by summation for j running from n to ∞. For instance, the convergence of the generalized $p-$series, with $p > 1$ can be proved like this:

$$\sum_{j=2}^{\infty} \frac{1}{j^p} < \int_1^{\infty} \frac{1}{x^p} dx = \frac{1}{p-1} < \infty.$$

On the other hand, if $p \leq 1$,

$$\sum_{j=1}^{\infty} \frac{1}{j^p} > \int_1^{\infty} \frac{1}{x^p} dx$$

and the divergence of the integral leads to the divergence of the $p-$series, too (consider the cases $p = 1$ and $p < 1$ separately to compute the integral – which is ∞ in either case).

Solution 2. This is the simplest of all proofs. Since

$$\begin{aligned}\frac{2}{\sqrt{k}}-\frac{2}{\sqrt{k+1}}&=\frac{2}{\sqrt{k}\sqrt{k+1}(\sqrt{k}+\sqrt{k+1})}\\&>\frac{2}{\sqrt{k}\sqrt{k+1}(\sqrt{k+1}+\sqrt{k+1})}\\&=\frac{1}{(k+1)\sqrt{k}},\end{aligned}$$

we have

$$\sum_{k=1}^{\infty}\frac{1}{(k+1)\sqrt{k}}<\sum_{k=1}^{\infty}\left(\frac{2}{\sqrt{k}}-\frac{2}{\sqrt{k+1}}\right)=2.$$

M38. *Remember the identity from the Example 3.5 and use it to prove that*

$$\sum_{k=1}^{\infty}\frac{1}{k^2}=\frac{\pi^2}{6}.$$

Solution. This is the celebrated Basel problem, solved by Euler at the age of 28. We have already seen in Example 3.5 that, for a positive integer n, the numbers

$$\cot^2\frac{k\pi}{2n+1},\ k=1,2,\ldots,n$$

are precisely the roots of the equation

$$\binom{2n+1}{1}x^n-\binom{2n+1}{3}x^{n-1}+\cdots+(-1)^n\binom{2n+1}{2n+1}=0.$$

(Remember that this comes from the formula – obtained from de Moivre's formula, and the binomial development – for the sine of a multiple of an angle. Namely, we have

$$\sin mt=\binom{m}{1}\cos^{m-1}t\sin t-\binom{m}{3}\cos^{m-3}t\sin^3 t+\cdots$$

thus, for $\sin t \neq 0$,

$$\frac{\sin mt}{\sin^m t} = \binom{m}{1}\cot^{m-1} t - \binom{m}{3}\cot^{m-3} t + \cdots .$$

Just put here $m = 2n+1$ odd, and $t = k\pi/(2n+1)$ in order to see that the n distinct numbers $\cot(k\pi/(2n+1))$, $k = 1, 2, \ldots, n$ are the roots of the above equation.)

Thus, their sum is

$$\sum_{k=1}^{n} \cot^2 \frac{k\pi}{2n+1} = \frac{\binom{2n+1}{3}}{\binom{2n+1}{1}} = \frac{n(2n-1)}{3}.$$

Also, we have

$$\sum_{k=1}^{n} \frac{1}{\sin^2 \dfrac{k\pi}{2n+1}} = \sum_{k=1}^{n}\left(1 + \cot^2 \frac{k\pi}{2n+1}\right) = n + \frac{n(2n-1)}{3} = \frac{n(2n+2)}{3}.$$

Now, the well-known inequalities $\sin x < x < \tan x$ imply

$$\cot^2 x < \frac{1}{x^2} < \frac{1}{\sin^2 x}$$

for all $x \in (0, \pi/2)$. Putting all these together yields

$$\sum_{k=1}^{n} \cot^2 \frac{k\pi}{2n+1} < \sum_{k=1}^{n} \frac{1}{\left(\dfrac{k\pi}{2n+1}\right)^2} < \sum_{k=1}^{n} \frac{1}{\sin^2 \dfrac{k\pi}{2n+1}},$$

or

$$\frac{\pi^2}{6} \cdot \frac{2n(2n-1)}{(2n+1)^2} < \sum_{k=1}^{n} \frac{1}{k^2} < \frac{\pi^2}{6} \cdot \frac{2n(2n+2)}{(2n+1)^2},$$

for every positive integer n. All that remains to do is passing to the limit for n tending to infinity, and use the squeeze theorem after noting that both extreme sides tend to the same limit, $\pi^2/6$.

Alternatively, we can use the formula

$$\sum_{k=1}^{n} \cot^2 \frac{(2k-1)\pi}{4n} = n(2n-1)$$

(how can it be obtained?) and the same inequalities as before in order to get

$$\frac{\pi^2}{8} \cdot \frac{2n-1}{2n} < \sum_{k=1}^{n} \frac{1}{(2k-1)^2} < \frac{\pi^2}{8}.$$

Passing to the limit yields

$$\sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} = \frac{\pi^2}{8}.$$

Basically these is the same as the Basel problem's formula, because

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} + \sum_{k=1}^{\infty} \frac{1}{(2k)^2} = \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} + \frac{1}{4}\sum_{k=1}^{\infty} \frac{1}{k^2},$$

hence

$$\sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} = \frac{3}{4}\sum_{k=1}^{\infty} \frac{1}{k^2}.$$

We invite the reader to follow carefully this path, too.

M39. *Evaluate*

(a) $\displaystyle\sum_{k=1}^{\infty} \frac{1}{1^3+2^3+\cdots+k^3}.$

(b) $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k}{1^3+2^3+\cdots+k^3}.$

Solution. (a) As we know, we have

$$1^3+2^3+\cdots+k^3 = \left(\frac{k(k+1)}{2}\right)^2,$$

hence

$$\begin{aligned}\sum_{k=1}^{n}\frac{1}{1^3+2^3+\cdots+k^3} &= 4\sum_{k=1}^{n}\left(\frac{1}{k}-\frac{1}{k+1}\right)^2\\ &= 4\sum_{k=1}^{n}\left(\frac{1}{k^2}+\frac{1}{(k+1)^2}-\frac{2}{k(k+1)}\right)\\ &= 4\left(2\sum_{k=1}^{n}\frac{1}{k^2}-1+\frac{1}{(n+1)^2}-2\left(1-\frac{1}{n+1}\right)\right),\end{aligned}$$

so that

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{1}{1^3+2^3+\cdots+k^3} &= \lim_{n\to\infty} 4\left(2\sum_{k=1}^{n}\frac{1}{k^2}-1+\frac{1}{(n+1)^2}-2\left(1-\frac{1}{n+1}\right)\right)\\ &= 4\left(\frac{\pi^2}{3}-3\right).\end{aligned}$$

We used the famous result of the previous problem stating that

$$\sum_{k=1}^{\infty}\frac{1}{k^2}=\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{k^2}=\frac{\pi^2}{6}.$$

(b) Similarly, we have

$$\begin{aligned}\sum_{k=1}^{n}\frac{(-1)^k}{1^3+2^3+\cdots+k^3} &= 4\sum_{k=1}^{n}(-1)^k\left(\frac{1}{k}-\frac{1}{k+1}\right)^2\\ &= 4\sum_{k=1}^{n}(-1)^k\left(\frac{1}{k^2}+\frac{1}{(k+1)^2}-\frac{2}{k(k+1)}\right)\\ &= 4\left(-1+\frac{(-1)^n}{(n+1)^2}-4\sum_{k=1}^{n}\frac{(-1)^k}{k}+2\cdot(-1)-2\frac{(-1)^{n+1}}{n+1}\right)\end{aligned}$$

because

$$\sum_{k=1}^{n}\frac{(-1)^k}{k(k+1)}=\sum_{k=1}^{n}\frac{(-1)^k}{k}-\sum_{k=1}^{n}\frac{(-1)^k}{k+1}=\sum_{k=1}^{n}\frac{(-1)^k}{k}+\sum_{k=1}^{n}\frac{(-1)^{k+1}}{k+1}$$

$$= 2\sum_{k=1}^{n}\frac{(-1)^k}{k} - (-1) + \frac{(-1)^{n+1}}{n+1}.$$

By passing to the limit, we get

$$\sum_{k=1}^{\infty}\frac{(-1)^k}{1^3+2^3+\cdots+k^3} = 4(4\ln 2 - 3)$$

if we use the well-known result

$$\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k} = \ln 2 \Leftrightarrow \sum_{k=1}^{n}\frac{(-1)^k}{k} = -\ln 2.$$

We saw this at the start of Chapter 6, by recognizing a generating function. Here is a sketch of another possible proof. We have

$$\begin{aligned}\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k} &= \sum_{k=1}^{n}(-1)^{k-1}\int_0^1 x^{k-1}dx = \int_0^1\sum_{k=1}^{n}(-x)^{k-1}dx\\ &= \int_0^1\frac{1-(-x)^n}{1+x}dx = \int_0^1\frac{1}{1+x}dx - \int_0^1\frac{(-x)^n}{1+x}dx\\ &= \ln 2 - \int_0^1\frac{(-x)^n}{1+x}dx.\end{aligned}$$

We let the reader show that the last integral has limit 0 for n tending to infinity (just notice that its absolute value is bounded from above by $\displaystyle\int_0^1 x^n dx$, which tends to 0), and thus prove the claimed result.

M40. *Let T be the set of all triples (a,b,c) of positive integers such that a,b,c are the lengths of the sides of some triangle. Evaluate*

$$\sum_{(a,b,c)\in T}\frac{2^a}{3^b5^c}.$$

Solution. It is well-known that a, b, and c are the sides of a triangle if and only if there exist positive k, l, and m such that

$$a = \frac{l+m}{2},\ b = \frac{k+m}{2},\ \text{ and } c = \frac{k+l}{2}.$$

In fact we have

$$k = b + c - a, \; l = a + c - b, \text{ and } m = a + b - c,$$

so that, if we need a, b, c to be positive integers, k, l, m must also be positive integers, and they must have the same parity. Consequently we have either $k = 2p - 1$, $l = 2q - 1$, and $m = 2r - 1$ (thus $a = q + r - 1$, $b = p + r - 1$, and $c = p + q - 1$), or $k = 2p$, $l = 2q$, and $m = 2r$ (hence $a = q + r$, $b = p + r$, and $c = p + q$), in both cases with positive integers p, q, and r. Accordingly, we have

$$\begin{aligned}
\sum_{(a,b,c)\in T} \frac{2^a}{3^b 5^c} &= \sum_{p,q,r\ge 1} \frac{2^{q+r-1}}{3^{p+r-1}5^{p+q-1}} + \sum_{p,q,r\ge 1} \frac{2^{q+r}}{3^{p+r}5^{p+q}} \\
&= \left(\frac{2^{-1}}{3^{-1}5^{-1}} + 1\right) \sum_{p,q,r\ge 1} \frac{2^{q+r}}{3^{p+r}5^{p+q}} \\
&= \frac{17}{2} \sum_{p,q,r\ge 1} \left(\frac{1}{15}\right)^p \left(\frac{2}{5}\right)^q \left(\frac{2}{3}\right)^r \\
&= \frac{17}{2} \sum_{p\ge 1} \left(\frac{1}{15}\right)^p \sum_{q\ge 1} \left(\frac{2}{5}\right)^q \sum_{r\ge 1} \left(\frac{2}{3}\right)^r \\
&= \frac{17}{2} \cdot \frac{\frac{1}{15}}{1 - \frac{1}{15}} \cdot \frac{\frac{2}{5}}{1 - \frac{2}{5}} \cdot \frac{\frac{2}{3}}{1 - \frac{2}{3}} = \frac{17}{21}.
\end{aligned}$$

Of course, when we wrote $\sum\limits_{p,q,r\ge 1}$ we meant that the sum is over all possible triples (p, q, r) of positive integers – that is why it can be expressed as the product of the three geometric series, each over all positive integers. The problem was in the Putnam Competition in the year 2015.

M41. *Prove that the inequality*

$$\left(\frac{a_1}{a_2}\right)^{n-1} + \left(\frac{a_2}{a_3}\right)^{n-1} + \cdots + \left(\frac{a_n}{a_1}\right)^{n-1} \ge 2\frac{a_1 + a_2 + \cdots + a_n}{\sqrt[n]{a_1 a_2 \cdots a_n}} - n$$

holds for any positive real numbers $a_1, a_2, \ldots, a_n$.

Solution. We prefer to denote $x_1 = \sqrt[n]{a_1}, x_2 = \sqrt[n]{a_2}, \ldots, x_n = \sqrt[n]{a_n}$ and thus to prove that

$$\left(\frac{x_1}{x_2}\right)^{n(n-1)} + \left(\frac{x_2}{x_3}\right)^{n(n-1)} + \cdots + \left(\frac{x_n}{x_1}\right)^{n(n-1)} + n \geq 2\frac{x_1^n + x_2^n + \cdots + x_n^n}{x_1 x_2 \cdots x_n},$$

for any positive $x_1, x_2, \ldots, x_n$.

We use the inequality between the arithmetic mean and the geometric mean of $n(n-1)$ positive numbers to get

$$\begin{aligned}(n-1)\left(\frac{x_1}{x_2}\right)^{n(n-1)} &+ (n-2)\left(\frac{x_2}{x_3}\right)^{n(n-1)} + \cdots + \left(\frac{x_{n-1}}{x_n}\right)^{n(n-1)} + \frac{n(n-1)}{2} \\ &\geq n(n-1) \sqrt[n(n-1)]{\left[\left(\frac{x_1}{x_2}\right)^{n-1}\left(\frac{x_2}{x_3}\right)^{n-2}\cdots\left(\frac{x_{n-1}}{x_n}\right)\right]^{n(n-1)}} \\ &= n(n-1)\frac{x_1^n}{x_1 x_2 \cdots x_n}\end{aligned}$$

($\frac{n(n-1)}{2}$ of the numbers are equal to 1). This yields, by cyclic permutations of the variables, $n-1$ more similar inequalities . Now one can immediately obtain the required inequality by adding up all these n inequalities (of course, after dividing the result by $\frac{n(n-1)}{2}$).

This is problem 11193 from *The American Mathematical Monthly*, December 2005. A solution by Koopa Koo, based on the same ideas, can be found in the 2017 August-September issue of the same *Monthly*.

3 Solutions to Hard Problems

H1. *Find all positive integers n for which*

$$N = \left(1^4 + \frac{1}{4}\right)\left(2^4 + \frac{1}{4}\right)\cdots\left(n^4 + \frac{1}{4}\right)$$

is the square of a rational number.

Solution. Suppose $n \geq 2$ (since for $n = 1$ one sees directly that $N = 5/4$ is not a square). If N is the square of a rational number, the same is true for

$$\begin{aligned}(2^n)^2 N = 4^n N &= \prod_{k=1}^{n}(4k^4 + 1) \\ &= \prod_{k=1}^{n}(2k^2 - 2k + 1)(2k^2 + 2k + 1) \\ &= \prod_{k=1}^{n}(2k^2 - 2k + 1)\left(2(k+1)^2 - 2(k+1) + 1\right) \\ &= \left(\prod_{k=2}^{n}(2k^2 - 2k + 1)\right)^2 (2n^2 + 2n + 1).\end{aligned}$$

(We have seen this trick before, haven't we?).
From N being a square it follows that $2n^2 + 2n + 1$ must be the square of a rational number, too, and, because $2n^2 + 2n + 1$ is a natural number, it actually must be the square of a natural number.
Conversely, if $2n^2 + 2n + 1$ is the square of a natural number, then $4^n N$ is the square of a natural number and, consequently, N is the square of a rational number. So, basically, the numbers that fulfil the condition from the statement of our problem are those n for which there is some natural number m such that $2n^2 + 2n + 1 = m^2$. For instance, $n = 3$ is such a solution, for which $N = (5^2 \cdot 13/8)^2$.

Now we can see that the equality $2n^2 + 2n + 1 = m^2$ is equivalent to $(2n + 1)^2 - 2m^2 = -1$, hence any solution (x, y) of the Pell type

equation $x^2 - 2y^2 = -1$ produces a solution (n, m) of the equation $2n^2 + 2n + 1 = m^2$ if we take $n = (x-1)/2$ and $m = y$ (clearly, x must be odd). From the theory of the Pell equations it is known that the solutions for $x^2 - 2y^2 = -1$ are given by the formulae

$$x = \frac{1}{2}\left((1+\sqrt{2})^{2q+1} + (1-\sqrt{2})^{2q+1}\right),$$
$$y = \frac{1}{2\sqrt{2}}\left((1+\sqrt{2})^{2q+1} - (1-\sqrt{2})^{2q+1}\right), \quad q \in \mathbb{N}.$$

So, finally, the numbers n that we are looking for are those of the form

$$n = \frac{1}{4}\left((1+\sqrt{2})^{2q+1} + (1-\sqrt{2})^{2q+1} - 2\right),$$

where q is any positive integer ($q = 0$ yields $n = 0$, too, which is not acceptable). Thus the first such n is 3 (as we already said), and the next one is $n = 20$.

H2. *Let $a_0 \geq 2$ and $a_{n+1} = a_n^2 - a_n + 1$, $n \geq 0$. Prove that*

$$\log_{a_0}(a_n - 1)\log_{a_1}(a_n - 1)\cdots\log_{a_{n-1}}(a_n - 1) \geq n^n$$

for all $n \geq 1$.

Solution. If $a_0 \geq 2$, we also have $a_0 > 1$, and an immediate induction shows that $a_n > 1$ for all n, hence the logarithms are well-defined. Also, $\log a_n > 0$ for any n, if log denotes any logarithm with basis greater than 1 (for example, the natural logarithm). Thus, the inequality to prove can be rearranged as

$$\frac{\log(a_n - 1)}{\log a_0}\frac{\log(a_n - 1)}{\log a_1}\cdots\frac{\log(a_n - 1)}{\log a_{n-1}} \geq n^n$$
$$\Leftrightarrow (\log a_0 \log a_1 \cdots \log a_{n-1})^{\frac{1}{n}} \leq \frac{1}{n}\log(a_n - 1).$$

By the inequality between the arithmetic and geometric means for the positive numbers $\log a_0, \log a_1, \ldots, \log a_{n-1}$, we have

$$\begin{aligned}(\log a_0 \log a_1 \cdots \log a_{n-1})^{\frac{1}{n}} &\leq \frac{1}{n}(\log a_0 + \log a_1 + \cdots + \log a_{n-1}) \\ &= \frac{1}{n}\log(a_0 a_1 \cdots a_{n-1}) \leq \frac{1}{n}\log(a_n - 1).\end{aligned}$$

So, we have our problem solved if we prove the last inequality, which is equivalent to $a_0a_1\cdots a_{n-1} \le a_n - 1$. But the recurrence relation for the sequence $(a_n)_{n\ge 0}$ can also be written in the form

$$a_k = \frac{a_{k+1}-1}{a_k-1},\ k = 0, 1, \ldots,$$

and, if we multiply these equalities for k running from 0 to $n-1$, we get

$$a_0a_1\cdots a_{n-1} = \frac{a_n-1}{a_0-1} \le a_n - 1,$$

because $a_0 - 1 \ge 1$, by hypothesis, which is precisely what we intended to prove.

H3. *Let a be a real number greater than 1. Evaluate*

$$\sum_{k=1}^{\infty}(-1)^{k-1}\frac{2^{k-1}a^{2^{k-1}-1}}{a^{2^k}-a^{2^{k-1}}+1}.$$

Solution. By repeatedly using

$$\frac{1}{a^2+a+1} - \frac{1}{a^2-a+1} = \frac{-2a}{a^4+a^2+1}$$

(you remember $(a^2-a+1)(a^2+a+1) = a^4+a^2+1$, don't you?), or by inducting on n, we get

$$\frac{1}{a^2+a+1} - \sum_{k=1}^{n}(-1)^{k-1}\frac{2^{k-1}a^{2^{k-1}-1}}{a^{2^k}-a^{2^{k-1}}+1} = (-1)^n\frac{2^n a^{2^n-1}}{a^{2^{n+1}}+a^{2^n}+1}.$$

Because for $a > 1$ we have

$$\lim_{n\to\infty}\frac{na^{n-1}}{a^{2n}+a^n+1} = \lim_{n\to\infty}\frac{\dfrac{n}{a^{n+1}}}{1+\dfrac{1}{a^n}+\dfrac{1}{a^{2n}}} = 0,$$

the right-hand side of the previous identity also has limit 0 when $n \to \infty$.

Consequently, the required sum is

$$\sum_{k=1}^{\infty}(-1)^{k-1}\frac{2^{k-1}a^{2^{k-1}-1}}{a^{2^k}-a^{2^{k-1}}+1} = \lim_{n\to\infty}\sum_{k=1}^{n}(-1)^{k-1}\frac{2^{k-1}a^{2^{k-1}-1}}{a^{2^k}-a^{2^{k-1}}+1}$$
$$= \frac{1}{a^2+a+1}.$$

H4. *For a nonnegative integer k, define $S_k(n) = 1^k + 2^k + \cdots + n^k$. Prove that*

$$1 + \sum_{k=0}^{r-1}\binom{r}{k}S_k(n) = (n+1)^r.$$

Solution. This is just a reminder (we did it before at the start of Chapter 1). We have

$$\sum_{k=0}^{r-1}\binom{r}{k}S_k(n) = \sum_{k=0}^{r-1}\binom{r}{k}\sum_{j=1}^{n}j^k = \sum_{j=1}^{k}\sum_{k=0}^{r-1}\binom{r}{k}j^k$$
$$= \sum_{j=1}^{k}\left((j+1)^r - j^r\right) = (n+1)^r - 1,$$

as required. The following recurrence for the sums $S_k(n)$ also holds:

$$S_{r+1}(n) - \sigma_1^r S_r(n) + \sigma_2^r S_{r-1}(n) - \cdots + (-1)^r\sigma_r^r S_1(n)$$
$$= \frac{(n+1)n(n-1)\cdots(n-r)}{r+2},$$

for positive integers r and n, where σ_k^r is the sum of all possible products of k distinct factors chosen from the numbers $1, 2, \ldots, r$ (the kth fundamental symmetric sum of the numbers $1, 2, \ldots, r$), for $1 \le k \le r$. Indeed,

$$S_{r+1}(n) - \sigma_1^r S_r(n) + \sigma_2^r S_{r-1}(n) - \cdots + (-1)^r\sigma_r^r S_1(n)$$
$$= \sum_{j=1}^{n}j^{r+1} - \sigma_1^r\sum_{j=1}^{n}j^r + \sigma_2^r\sum_{j=1}^{n}j^{r-1} - \cdots + (-1)^r\sum_{j=1}^{n}j$$

$$= \sum_{j=1}^{n} \left(j^{r+1} - \sigma_1^r j^r + \sigma_2^r j^{r-1} - \cdots + (-1)^r \sigma_r^r j\right) = \sum_{j=1}^{n} j(j-1)\cdots(j-r)$$

$$= \frac{1}{r+2} \sum_{j=1}^{n} ((j+1)j\cdots(j-r) - j\cdots(j-r)(j-r-1))$$

$$= \frac{(n+1)n(n-1)\cdots(n-r)}{r+2}.$$

Get a slightly different recurrence by using the similar formula

$$j^{r+1} + \sigma_1^r j^r + \sigma_2^r j^{r-1} + \cdots + \sigma_r^r j = j(j+1)\cdots(j+r).$$

For instance, when $r = 2$, we have $\sigma_1^2 = 1 + 2 = 3$ and $\sigma_2^2 = 1 \cdot 2 = 2$, therefore we get

$$S_3(n) - 3S_2(n) + 2S_1(n) = \frac{(n+1)n(n-1)(n-2)}{4}.$$

By replacing here $S_1(n) = n(n+1)/2$ and $S_2(n) = n(n+1)(2n+1)/6$, we get $S_3(n) = (n(n+1)/2)^2$, as we know. Use this recurrence formula to find $S_4(n)$.

H5. *Find all positive integers n such that*

$$n = \prod_{i=0}^{m} (a_i + 1),$$

where $\overline{a_m a_{m-1} \ldots a_0}$ is the decimal representation of n.

Solution. For $m = 0$ there clearly are no solutions, so that we assume $m \geq 1$ (that is, n has at least two digits).
We have, for a solution $n = \overline{a_m a_{m-1} \ldots a_0}$,

$$\begin{aligned}\overline{a_m a_{m-1}} \cdot 10^{m-1} &\leq \overline{a_m a_{m-1} \ldots a_0} = (a_m + 1)(a_{m-1} + 1) \cdots (a_0 + 1)\\ &\leq (a_m + 1)(a_{m-1} + 1) \cdot 10^{m-1}.\end{aligned}$$

The first inequality comes from the fact that all digits are at least 0, and the second one is due to the fact that the digits are at most 9 (we use these bounds for the last $m-2$ digits). Consequently

$$10a_m + a_{m-1} = \overline{a_m a_{m-1}} \leq (a_m+1)(a_{m-1}+1),$$

which is equivalent to

$$(9-a_{m-1})a_m \leq 1.$$

Since both $9-a_{m-1}$ and a_m are from the set $\{0,1,\ldots,9\}$, the last inequality is possible for either $a_m = 1$ and $a_{m-1} = 8$, or $a_{m-1} = 9$ and a_m any other digit.

In the first case the above inequalities become

$$18 \cdot 10^{m-1} \leq \overline{18a_{m-2}\ldots a_0} = 2 \cdot 9 \cdot (a_{m-2}+1)\cdots(a_0+1) \leq 18 \cdot 10^{m-1}.$$

That is, they must be all equalities, which is possible if and only if $a_{m-2}, \ldots, a_0$ are all 0, and, simultaneously, they are all equal to 9. Of course, this can only happen when $m = 1$ (so there are no $a_{m-2}, \ldots, a_0$) – that is we get the solution $n = 18$.

In the second case the equality

$$\overline{a_m a_{m-1}\ldots a_0} = (a_m+1)\cdots(a_0+1)$$

forces a_0 to be 0 (because there is a factor of 10 in the right-hand side), which makes the right-hand side at most equal to 10^m. On the other hand, the left hand side is at least $19 \cdot 10^{m-1}$, as $a_m \geq 1$, and $a_{m-1} = 9$. Since $19 \cdot 10^{m-1} > 10^m$ the equality is not possible in this case. The only solution remains the number 18.

H6. *Let*

$$a_k = \frac{k}{(k-1)^{\frac{4}{3}} + k^{\frac{4}{3}} + (k+1)^{\frac{4}{3}}}.$$

Prove that $a_1 + a_2 + \cdots + a_{999} < 50$.

Solution. We have

$$\begin{aligned}
\sum_{k=1}^{999} a_k &= \sum_{k=1}^{999} \frac{k}{(k-1)^{\frac{4}{3}} + k^{\frac{4}{3}} + (k+1)^{\frac{4}{3}}} \\
&< \sum_{k=1}^{999} \frac{k}{(k-1)^{\frac{4}{3}} + (k^2-1)^{\frac{2}{3}} + (k+1)^{\frac{4}{3}}} \\
&= \sum_{k=1}^{999} \frac{k\left((k+1)^{\frac{2}{3}} - (k-1)^{\frac{2}{3}}\right)}{(k+1)^2 - (k-1)^2} \\
&= \frac{1}{4} \sum_{k=1}^{999} \left((k+1)^{\frac{2}{3}} - (k-1)^{\frac{2}{3}}\right) \\
&= \frac{1}{4}\left(1000^{\frac{2}{3}} + 999^{\frac{2}{3}} - 1^{\frac{2}{3}} - 0^{\frac{2}{3}}\right) \\
&< \frac{1}{4}(100 + 100 - 1) < 50.
\end{aligned}$$

H7. *For a fixed positive integer a define the sequence*

$$a_n = \left\lfloor \left(a + \sqrt{a^2+1}\right)^n + \left(\frac{1}{2}\right)^n \right\rfloor, \quad n \geq 0.$$

Prove that

$$\sum_{n=1}^{\infty} \frac{1}{a_{n-1}a_{n+1}} = \frac{1}{8a^2}.$$

Solution. We show first that

$$a_n = \left(a + \sqrt{a^2+1}\right)^n + \left(a - \sqrt{a^2+1}\right)^n$$

for every $n \geq 0$. Note first that, by the binomial theorem, the number

$$\begin{aligned}
b_n &= \left(a + \sqrt{a^2+1}\right)^n + \left(a - \sqrt{a^2+1}\right)^n \\
&= \sum_{k=0}^{\lfloor n/2 \rfloor} 2\binom{n}{2k} a^{n-2k}(a^2+1)^k
\end{aligned}$$

is an integer, for each $n \geq 0$. Thus we have

$$\begin{aligned} a_n &= \left\lfloor \left(a+\sqrt{a^2+1}\right)^n + \left(\frac{1}{2}\right)^n \right\rfloor \\ &= b_n + \left\lfloor -\left(a-\sqrt{a^2+1}\right)^n + \left(\frac{1}{2}\right)^n \right\rfloor \\ &= b_n, \end{aligned}$$

if we prove that the last integral part is 0. Observe that

$$\sqrt{a^2+1} - a = \frac{1}{\sqrt{a^2+1}+a} < \frac{1}{2}$$

for $n \geq 0$, as $a \geq 1$ and $\sqrt{a^2+1} > 1$. Now we have, for odd n,

$$-\left(a-\sqrt{a^2+1}\right)^n + \left(\frac{1}{2}\right)^n = \left(\sqrt{a^2+1}-a\right)^n + \left(\frac{1}{2}\right)^n < 2 \cdot \left(\frac{1}{2}\right)^n \leq 1$$

and

$$-\left(a-\sqrt{a^2+1}\right)^n + \left(\frac{1}{2}\right)^n = \left(\sqrt{a^2+1}-a\right)^n + \left(\frac{1}{2}\right)^n > 0.$$

For even n we have

$$-\left(a-\sqrt{a^2+1}\right)^n + \left(\frac{1}{2}\right)^n = \left(\frac{1}{2}\right)^n - \left(\sqrt{a^2+1}-a\right)^n < \left(\frac{1}{2}\right)^n \leq 1$$

and, also,

$$-\left(a-\sqrt{a^2+1}\right)^n + \left(\frac{1}{2}\right)^n = \left(\frac{1}{2}\right)^n - \left(\sqrt{a^2+1}-a\right)^n > 0.$$

Thus in both cases the expression $-\left(a-\sqrt{a^2+1}\right)^n + \left(\frac{1}{2}\right)^n$ is between 0 and 1, therefore its floor function is 0. So,

$$a_n = \left(a+\sqrt{a^2+1}\right)^n + \left(a-\sqrt{a^2+1}\right)^n$$

for every $n \geq 0$. Because $\left(a+\sqrt{a^2+1}\right)^n$ and $\left(a-\sqrt{a^2+1}\right)^n$ are the roots of the quadratic equation $x^2 - 2ax - 1 = 0$, it follows that the numbers a_n satisfy the recurrence

$$a_{n+1} - 2aa_n - a_{n-1} = 0 \Leftrightarrow a_{n+1} - a_{n-1} = 2aa_n$$

for $n \geq 1$, hence

$$\sum_{n=1}^{\infty} \frac{1}{a_{n-1}a_{n+1}} = \frac{1}{2a}\sum_{n=1}^{\infty} \frac{2aa_n}{a_{n-1}a_na_{n+1}} = \frac{1}{2a}\sum_{n=1}^{\infty} \frac{a_{n+1}-a_{n-1}}{a_{n-1}a_na_{n+1}}$$
$$= \frac{1}{2a}\sum_{n=1}^{\infty}\left(\frac{1}{a_{n-1}a_n} - \frac{1}{a_na_{n+1}}\right) = \frac{1}{2a}\frac{1}{a_0a_1} = \frac{1}{8a^2},$$

because $a_0 = 2$ and $a_1 = 2a$.

H8. *Evaluate*

$$\sum_{k=1}^{\infty} \frac{1}{2^k} \tan \frac{a}{2^k},$$

where $a \neq s\pi$, with s any integer.

Solution. Remember (and prove!) the formula

$$\tan x = \cot x - 2\cot 2x,$$

according to which we can write

$$\sum_{k=1}^{n} \frac{1}{2^k}\tan\frac{a}{2^k} = \sum_{k=1}^{n}\left(\frac{1}{2^k}\cot\frac{a}{2^k} - \frac{1}{2^{k-1}}\cot\frac{a}{2^{k-1}}\right) = \frac{1}{2^n}\cot\frac{a}{2^n} - \cot a.$$

Consequently,

$$\sum_{k=1}^{\infty} \frac{1}{2^k}\tan\frac{a}{2^k} = \lim_{n\to\infty}\left(\frac{1}{2^n}\cot\frac{a}{2^n} - \cot a\right) = \frac{1}{a} - \cot a.$$

This is because we have the well-known limit

$$\lim_{x\to 0} x\cot x = \lim_{x\to 0}\frac{x}{\tan x} = 1$$

yielding

$$\lim_{n\to\infty} \frac{1}{2^n} \cot \frac{a}{2^n} = \lim_{n\to\infty} \frac{1}{a} \cdot \frac{a}{2^n} \cot \frac{a}{2^n} = \frac{1}{a}$$

(as $a/2^n \to 0$ when $n \to \infty$).

H9. *Let n be a positive integer. Prove that*

$$\prod_{k=0}^{n-1} \left(2\sin^2 \frac{(k-1)\pi}{n} + 2\sin^2 \frac{(k+1)\pi}{n} - \sin^2 \frac{2\pi}{n} \right) = \left(1 - \cos^n \frac{2\pi}{n} \right)^2.$$

Solution. As we know, we have the factorization

$$z^n - 1 = \prod_{k=0}^{n-1} \left(z - \cos \frac{2k\pi}{n} - i \sin \frac{2k\pi}{n} \right)$$

for any complex number z. Taking the absolute values, we get

$$|z^n - 1|^2 = \prod_{k=0}^{n-1} \left| z - \cos \frac{2k\pi}{n} - i \sin \frac{2k\pi}{n} \right|^2,$$

hence

$$|z^n - 1|^2 = \prod_{k=0}^{n-1} \left(z^2 - 2z \cos \frac{2k\pi}{n} + 1 \right),$$

when z is a real number.

For $z = \cos \frac{2\pi}{n}$ we get

$$\begin{aligned}
\left(1 - \cos^n \frac{2\pi}{n} \right)^2 &= \prod_{k=0}^{n-1} \left(\cos^2 \frac{2\pi}{n} - 2\cos \frac{2\pi}{n} \cos \frac{2k\pi}{n} + 1 \right) \\
&= \prod_{k=0}^{n-1} \left(\cos^2 \frac{2\pi}{n} - \cos \frac{2(k-1)\pi}{n} - \cos \frac{2(k+1)\pi}{n} + 1 \right) \\
&= \prod_{k=0}^{n-1} \left(2\sin^2 \frac{(k-1)\pi}{n} + 2\sin^2 \frac{(k+1)\pi}{n} - \sin^2 \frac{2\pi}{n} \right)
\end{aligned}$$

after a few simple trigonometric transforms, as desired.

H10. *Let m and n be integers greater than 1. Prove that*

$$\sum_{k_1+k_2+\cdots+k_n=m, k_1,k_2,\ldots,k_n\geq 0} \frac{1}{k_1!k_2!\cdots k_n!}\cos\left((k_1+2k_2+\cdots+nk_n)\frac{2\pi}{n}\right)=0.$$

Solution. The sum is over all n-tuples $(k_1, k_2, \ldots, k_n)$ of nonnegative integers that sum to m and we first observe that it represents the real part of

$$\sum_{k_1+k_2+\cdots+k_n=m, k_1,k_2,\ldots,k_n\geq 0} \frac{1}{k_1!k_2!\cdots k_n!} e^{i(k_1+2k_2+\cdots+nk_n)\frac{2\pi}{n}}$$

$$= \sum_{k_1+k_2+\cdots+k_n=m, k_1,k_2,\ldots,k_n\geq 0} \frac{1}{k_1!k_2!\cdots k_n!}\left(e^{\frac{2\pi i}{n}}\right)^{k_1}\left(e^{2\cdot\frac{2\pi i}{n}}\right)^{k_2}\cdots\left(e^{n\cdot\frac{2\pi i}{n}}\right)^{k_n}$$

$$= \frac{1}{m!}(1+\omega+\cdots+\omega^{n-1})^m$$

for $\omega = e^{\frac{2\pi i}{n}}$. Thus we arrived at the sum of the nth roots of unity, which is well-known to be 0, hence the conclusion follows (and we see that it remains true if the cosines are replaced by sines, too). We used the multinomial formula

$$\sum_{k_1+k_2+\cdots+k_n=m, k_1,k_2,\ldots,k_n\geq 0} \frac{m!}{k_1!k_2!\cdots k_n!} x_1^{k_1}x_2^{k_2}\cdots x_n^{k_n}=(x_1+x_2+\cdots+x_n)^m.$$

H11. *Let X be a set with n elements. Prove that*

$$\sum_{Y,Z\subseteq X} |Y\cap Z| = n\cdot 4^{n-1}.$$

The sum is over all possible pairs (Y, Z) of subsets of X.

Solution 1. First we note that if M is a finite set with m elements, then the number of pairs (A, B) of disjoint subsets of M is 3^m. Indeed, we can choose A having $k \leq m$ elements (from the m elements of M) in $\binom{m}{k}$ ways. Once we chose A, B can be any subset of the complement

$M \setminus A$ of A with respect to M. Since $M \setminus A$ has $m-k$ elements, there are 2^{m-k} possibilities for B (for every subset A of M with k elements). Thus the number of such pairs (A, B) is

$$\sum_{k=0}^{m} \binom{m}{k} 2^{m-k} = (2+1)^m = 3^m.$$

Now we evaluate the sum from the statement of our problem in the following way. For a given subset S of X, we can have $Y \cap Z = S$ for $Y, Z \subseteq X$ in as many ways as there are pairs of disjoint subsets of $X \setminus S$. (This is because $Y \cap Z = S$ is possible if and only if $Y = S \cup Y_1$ and $Z = S \cup Z_1$ for mutually disjoint Y_1 and Z_1 with elements outside S.) According to the observation from the beginning, we see that there are $3^{|X|-|S|}$ pairs (Y, Z) of subsets of X such that $Y \cap Z = S$. Of course, if S has $k \leq n$ elements, it can be chosen in $\binom{n}{k}$ ways, and it contributes to the sum with an amount of k. All these being said, we can calculate

$$\sum_{Y,Z \subset X} |Y \cap Z| = \sum_{k=0}^{n} k \binom{n}{k} 3^{n-k} = \sum_{k=1}^{n} k \binom{n}{k} 3^{n-k}.$$

As we have previously seen, by differentiating

$$\sum_{k=0}^{n} \binom{n}{k} x^k = (1+x)^n$$

(with respect to the variable x), we get

$$\sum_{k=1}^{n} k \binom{n}{k} x^{k-1} = n(1+x)^{n-1},$$

therefore we also have

$$\sum_{k=1}^{n} k \binom{n}{k} x^k = nx(1+x)^{n-1}.$$

For our sum we can write

$$\sum_{k=0}^{n} k\binom{n}{k}3^{n-k} = 3^n \sum_{k=1}^{n} k\binom{n}{k}\left(\frac{1}{3}\right)^k = 3^n \cdot n \cdot \frac{1}{3}\left(1+\frac{1}{3}\right)^{n-1} = n \cdot 4^{n-1},$$

which is precisely the claimed result. And here comes the simpler proof.

Solution 2.

$$\begin{aligned}\sum_{Y,Z\subseteq X} |Y \cap Z| &= \sum_{Y,Z\subseteq X}\ \sum_{x\in Y\cap Z} 1 \\ &= \sum_{x\in X}\sum_{x\in Y}\sum_{x\in Z} 1 = \sum_{x\in X} 2^{n-1}\cdot 2^{n-1} = n\cdot 4^{n-1},\end{aligned}$$

where we have used the fact that

$$\sum_{x\in Y} 1 = \sum_{x\in Z} 1 = 2^{n-1}$$

which just says in formulas that each $x \in X$ is in exactly 2^{n-1} subsets of X.

H12. *Evaluate the sum*

$$S_n = \binom{n}{1} - 3\binom{n}{3} + 5\binom{n}{5} - 7\binom{n}{7} + \cdots$$

Solution. Remember that

$$\begin{aligned}(1+i)^m = \sum_{k=0}^{m}\binom{m}{k}i^k &= \binom{m}{0} + \binom{m}{1}i - \binom{m}{2} - \binom{m}{3}i \\ &+ \binom{m}{4} + \binom{m}{5}i - \binom{m}{6} - \binom{m}{7}i + \cdots\end{aligned}$$

and

$$\begin{aligned}(1-i)^m = \sum_{k=0}^{m}\binom{m}{k}(-i)^k &= \binom{m}{0} - \binom{m}{1}i - \binom{m}{2} + \binom{m}{3}i \\ &+ \binom{m}{4} - \binom{m}{5}i - \binom{m}{6} + \binom{m}{7}i + \cdots,\end{aligned}$$

hence

$$(1+i)^m + (1-i)^m = 2\left(\binom{m}{0} - \binom{m}{2} + \binom{m}{4} - \binom{m}{6} + \cdots\right).$$

Since

$$(1 \pm i)^m = 2^{\frac{m}{2}}\left(\cos\frac{m\pi}{4} \pm i\sin\frac{m\pi}{4}\right),$$

we obtain

$$\sum_{j\ge 0}(-1)^j\binom{m}{2j} = \binom{m}{0} - \binom{m}{2} + \binom{m}{4} - \binom{m}{6} + \cdots = 2^{\frac{m}{2}}\cos\frac{m\pi}{4}.$$

Now remember the simple formula

$$j\binom{n}{j} = n\binom{n-1}{j-1}$$

and utilize all that to evaluate the given sum:

$$\begin{aligned} S_n &= \sum_{k\ge 1}(-1)^{k-1}(2k-1)\binom{n}{2k-1} = \sum_{k\ge 1}(-1)^{k-1}n\binom{n-1}{2k-2} \\ &= n\sum_{j\ge 0}(-1)^j\binom{n-1}{2j} = n2^{\frac{n-1}{2}}\cos\frac{(n-1)\pi}{4}. \end{aligned}$$

Of course, the sums are not infinite: at a certain moment the binomial coefficients become 0. For instance, by

$$\sum_{k\ge 1}(-1)^{k-1}(2k-1)\binom{n}{2k-1}$$

we actually mean

$$\sum_{k=1}^{\lfloor (n+1)/2\rfloor}(-1)^{k-1}(2k-1)\binom{n}{2k-1}$$

since the binomial coefficients with $2k-1 > n$ are all 0. The condition $2k-1 \le n$ and its analogues for other similar sums are somehow taken for granted in such wording.

H13. *Prove that*

$$\sum_{k\equiv 1 \pmod 3} \binom{n}{k} = \frac{1}{3}\left(2^n + 2\cos\left(\frac{(n-2)\pi}{3}\right)\right).$$

Solution. We already proved a general result on sums of this type (in Example 3.7), namely

$$\sum_{l\equiv 0 \pmod k} \binom{n}{l} = \binom{n}{0} + \binom{n}{k} + \binom{n}{2k} + \cdots = \frac{2^n}{k}\sum_{j=0}^{k-1} \cos^n \frac{j\pi}{k} \cos \frac{nj\pi}{k}.$$

Of course, there exists a more general formula, and this is

$$\begin{aligned}\sum_{l\equiv r \pmod k} \binom{n}{l} &= \binom{n}{r} + \binom{n}{r+k} + \binom{n}{r+2k} + \cdots \\ &= \frac{2^n}{k}\sum_{j=0}^{k-1} \cos^n \frac{j\pi}{k} \cos \frac{(n-2r)j\pi}{k}.\end{aligned}$$

(Again, the sums are finite.) Let us remember how we did in the case $r = 0$, by proving the general formula. We use the kth roos of unity,

$$\varepsilon_j = \cos\frac{2j\pi}{k} + i\sin\frac{2j\pi}{k},\ \ j = 0, 1, \ldots, k-1,$$

and the fact that their power sums with integer exponent t are

$$\sum_{j=0}^{k-1} \varepsilon_j^t = \begin{cases} 0 & \text{if} \quad k \nmid t \\ k & \text{if} \quad k \mid t. \end{cases}$$

Thus we have

$$\sum_{j=0}^{k-1} \varepsilon_j^{-r}(1+\varepsilon_j)^n = \sum_{j=0}^{k-1} \varepsilon_j^{-r} \sum_{s=0}^{n} \binom{n}{s} \varepsilon_j^s = \sum_{s=0}^{n} \binom{n}{s} \sum_{j=0}^{k-1} \varepsilon_j^{s-r}.$$

Now the inner sums are not 0 (in which case they equal k) precisely when $s \equiv r \pmod{k}$, therefore the above becomes

$$\sum_{j=0}^{k-1} \varepsilon_j^{-r}(1+\varepsilon_j)^n = k \sum_{l \equiv r \pmod{k}} \binom{n}{l}.$$

In order to obtain the desired formula we still need to note that, actually (being a real number), the sum from the right-hand side is equal to the real part of the sum from the left, and that, by using

$$1 + \cos\alpha + i\sin\alpha = 2\cos\frac{\alpha}{2}\left(\cos\frac{\alpha}{2} + i\sin\frac{\alpha}{2}\right)$$

plus de Moivre's formula and complex multiplication (in trigonometric form), we get

$$\varepsilon_j^{-r}(1+\varepsilon_j)^n = 2^n \cos^n\frac{j\pi}{k}\left(\cos\frac{(n-2r)j\pi}{k} + i\sin\frac{(n-2r)j\pi}{k}\right).$$

For instance, let us consider the case $k = 3$ when we have the three sums

$$S_r = \sum_{l \equiv r \pmod{3}} \binom{n}{l}, \quad r = 0, 1, 2.$$

The three roots of unity of order 3 are $\varepsilon_0 = 1$,

$$\varepsilon_1 = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}, \quad \text{and} \quad \varepsilon_2 = \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}.$$

By the binomial formula and the fact that $\varepsilon_j^3 = 1$,

$$S_0 + S_1 + S_2 = (1+1)^n = 2^n,$$

$$S_0 + \varepsilon_1 S_1 + \varepsilon_1^2 S_2 = (1+\varepsilon_1)^n,$$

and

$$S_0 + \varepsilon_2 S_1 + \varepsilon_2^2 S_2 = (1+\varepsilon_2)^n.$$

In order to solve for S_1 we need multiply the three equations with ε_0^{-1} (that is by 1), with ε_1^{-1}, and with ε_2^{-1} respectively then add and note that the coefficients of S_0 and S_2 become 0, while the coefficient of S_1 will be 3. Thus we get

$$3S_1 = 2^n + \varepsilon_1^{-1}(1+\varepsilon_1)^n + \varepsilon_2^{-1}(1+\varepsilon_2)^n,$$

and further, by trigonometric and complex numbers calculations,

$$3S_1 = 2^n + 2^n \cos^n \frac{\pi}{3} \cos \frac{(n-2)\pi}{3} + 2^n \cos^n \frac{2\pi}{3} \cos \frac{2(n-2)\pi}{3}.$$

If we take into account that

$$\begin{aligned}\cos \frac{2(n-2)\pi}{3} &= \cos\left((n-2)\pi - \frac{(n-2)\pi}{3}\right) = (-1)^{n-2} \cos \frac{(n-2)\pi}{3} \\ &= (-1)^n \cos \frac{(n-2)\pi}{3},\end{aligned}$$

and

$$\cos \frac{\pi}{3} = \frac{1}{2}, \ \cos \frac{2\pi}{3} = -\frac{1}{2},$$

we finally obtain the formula

$$S_1 = \binom{n}{1} + \binom{n}{4} + \binom{n}{7} + \cdots = \frac{1}{3}\left(2^n + 2\cos\frac{(n-2)\pi}{3}\right).$$

Similarly we can get

$$S_0 = \binom{n}{0} + \binom{n}{3} + \binom{n}{6} + \cdots = \frac{1}{3}\left(2^n + 2\cos\frac{n\pi}{3}\right)$$

and

$$S_2 = \binom{n}{2} + \binom{n}{5} + \binom{n}{8} + \cdots = \frac{1}{3}\left(2^n + 2\cos\frac{(n-4)\pi}{3}\right).$$

H14. *Prove that*

$$\sum_{k=0}^{m} \binom{n}{k}\binom{n}{m-k} = \binom{2n}{m}$$

for any nonnegative integers m and n.

Solution. If we want to choose an $m-$element subset of the set $\{1, 2, \ldots, 2n\}$ we can choose arbitrarily (for any $0 \leq k \leq m$) a $k-$element subset of $\{1, 2, \ldots, n\}$ (and this can be done in $\binom{n}{k}$ ways), then we complete with $m - k$ elements also arbitrarily chosen from $\{n+1, n+2, \ldots, 2n\}$ (which action can be performed in $\binom{n}{m-k}$ ways). So, there are $\sum\limits_{k=0}^{m} \binom{n}{k}\binom{n}{m-k}$ ways to chose m elements of the set $\{1, 2, \ldots, 2n\}$. On the other hand, $\binom{2n}{m}$ definitely counts the same thing. Thus, as both sides of the identity count the $m-$element subsets of $\{1, 2, \ldots, 2n\}$, they are, indeed, equal. Of course, this is a particular case of Vandermonde's identity

$$\sum_{i+j=c} \binom{a}{i}\binom{b}{j} = \binom{a+b}{c}$$

(a, b, and c are nonnegative integers and the sum is over all pairs (i, j) of nonnegative indices that sum to c), which we met (and proved) before. Here we provided the purely combinatorial approach and we invite the reader to extend the proof to the general case.

H15. *Let n be a positive integer. Prove the combinatorial identity*

$$\sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k} = \sum_{k=0}^{n} 2^k \binom{n}{k}^2.$$

Solution. Let $M = \{1, 2, \ldots, n\}$ and $N = \{n+1, n+2, \ldots, 2n\}$. We count the number of ordered pairs (X, Y) of subsets X of M and Y of $X \cup N$, Y having n elements. First, this number is $\sum\limits_{k=0}^{n} \binom{n}{k}\binom{n+k}{k}$, because, for each $0 \leq k \leq n$, we can choose a subset with k elements X of M in

$\binom{n}{k}$ ways, and once this X was selected, we can choose Y (a subset with n elements of $X \cup N$) in $\binom{n+k}{n} = \binom{n+k}{k}$ ways.

On the other hand, we can first pick up Y as a subset of $M \cup N$. More precisely, we first choose $Y \cap N$ as a subset of N that can have any number $k \le n$ of elements from the n elements of N – and this can be done in $\binom{n}{k}$ ways. The remaining $n-k$ elements of Y can be chosen from the n elements of M in $\binom{n}{n-k} = \binom{n}{k}$ ways, and for each of these choices, X can be completed with some of the other k elements of M (other than those already put in Y) in 2^k ways. Thus a pair (X, Y) of sets $X \subseteq M$ and $Y \subseteq X \cup N$ with $|Y| = n$ can be also chosen in $\sum_{k=0}^{n} 2^k \binom{n}{k}^2$ ways, and the equality of the two sums follows from this enumerative argument.

Prove similarly the slightly more general formula

$$\sum_{k=0}^{m} \binom{n}{k}\binom{n+k}{m} = \sum_{k=0}^{n} 2^k \binom{m}{k}\binom{n}{k}.$$

H16. *Let n be a positive integer. Prove that*

$$\sum_{k=1}^{n} \frac{(-1)^{k-1}}{k}\binom{n}{k} = 1 + \frac{1}{2} + \cdots + \frac{1}{n}.$$

Solution. We have

$$\frac{1}{k} = \int_0^1 x^{k-1}dx,$$

hence

$$1 + \frac{1}{2} + \cdots + \frac{1}{n} = \int_0^1 (1 + x + \cdots + x^{n-1})dx = \int_0^1 \frac{x^n - 1}{x - 1}dx$$
$$= \int_{-1}^0 \frac{(1+t)^n - 1}{t}dt,$$

where, at the last step, we changed the variable with $x = 1 + t$.

By developing $(1+t)^n$ with the binomial formula, and after simplifying

by t, we get

$$1+\frac{1}{2}+\cdots+\frac{1}{n}=\int_{-1}^{0}\sum_{k=1}^{n}\binom{n}{k}t^{k-1}=\sum_{k=1}^{n}\binom{n}{k}\left[\frac{t^k}{k}\right]_{-1}^{0}=\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k}\binom{n}{k},$$

as desired.

Solution 2. An elementary approach is possible, too. Let S_n denote the sum from the left-hand side. We have $S_1 = 1$ and (by the recurrence formula of binomial coefficients),

$$S_{n+1}=\sum_{k=1}^{n+1}\frac{(-1)^{k-1}}{k}\left(\binom{n}{k}+\binom{n}{k-1}\right).$$

Actually, for $k = n + 1$ the formula only gives

$$\binom{n+1}{n+1}=\binom{n}{n}(=1),\quad \text{as}\quad \binom{n}{n+1}=0,$$

so that we further can write

$$\begin{aligned}S_{n+1}&=\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k}\binom{n}{k}+\sum_{k=1}^{n+1}\frac{(-1)^{k-1}}{k}\binom{n}{k-1}\\&=S_n+\frac{1}{n+1}\sum_{k=1}^{n+1}(-1)^{k-1}\binom{n+1}{k}\\&=S_n+\frac{1}{n+1}.\end{aligned}$$

Now from $S_1 = 1$ and $S_{n+1} = S_n + 1/(n + 1)$ for $n \geq 1$ the conclusion

$$S_n=1+\frac{1}{2}+\cdots+\frac{1}{n}$$

easily follows by induction.

Note that we used

$$\frac{1}{k}\binom{n}{k-1}=\frac{1}{n+1}\binom{n+1}{k}\quad \text{and}\quad \sum_{k=0}^{n+1}(-1)^k\binom{n+1}{k}=0.$$

H17. *Let $(F_n)_{n\geq 0}$ be the Fibonacci sequence defined by*

$$F_0 = 0,\ F_1 = 1 \text{ and } F_{n+2} = F_{n+1} + F_n \text{ for } n \geq 0.$$

Prove that

$$\sum_{k\geq 0}\binom{n-k}{k} = F_{n+1}.$$

Obviously, the sum lasts as long as the binomial coefficient is not 0 *(that is, as long as $n-k \geq k$).*

Solution. If we denote by

$$S_n = \sum_{k\geq 0}\binom{n-k}{k},$$

we see immediately that $S_0 = 1$, $S_1 = 1$, and (by the binomial recurrence)

$$\begin{aligned} S_n &= \sum_{k\geq 0}\binom{n-k}{k} = \sum_{k\geq 0}\binom{n-k-1}{k} + \sum_{k\geq 1}\binom{n-k-1}{k-1} \\ &= \sum_{k\geq 0}\binom{n-1-k}{k} + \sum_{k\geq 1}\binom{n-2-(k-1)}{k-1} \\ &= S_{n-1} + S_{n-2} \end{aligned}$$

for all $n \geq 2$. Since $S_0 = F_1$, $S_1 = F_2$, and the sequences $(S_n)_{n\geq 0}$ and $(F_{n+1})_{n\geq 0}$ satisfy the same recurrence relation, they must coincide, that is, $S_n = F_{n+1}$ for every $n \geq 0$, as required.

H18. *Evaluate*

$$\binom{n}{0} - \binom{n-1}{1} + \binom{n-2}{2} - \binom{n-3}{3} + \cdots.$$

Solution. That is, we are asked to evaluate

$$\sum_{k\geq 0}(-1)^k\binom{n-k}{k},$$

which suggests us to consider the more general problem of evaluating the sum

$$S_n(x) = \sum_{k\geq 0} \binom{n-k}{k} x^k.$$

With this notation, we must determine $S_n(-1)$ (after we found, in the previous problem, that $S_n(1) = F_{n+1}$). A recurrence relation can be obtained exactly as in the preceding exercise:

$$\begin{aligned} S_n(x) &= \sum_{k\geq 0} \binom{n-k-1}{k} x^k + \sum_{k\geq 1} \binom{n-k-1}{k-1} x^k \\ &= \sum_{k\geq 0} \binom{n-1-k}{k} x^k + x \sum_{k\geq 1} \binom{n-2-(k-1)}{k-1} x^{k-1} \\ &= S_{n-1}(x) + xS_{n-2}(x) \end{aligned}$$

for all $n \geq 2$.

In the particular case of $x = -1$ we have $S_0(-1) = S_1(-1) = 1$, and the recurrence

$$S_n(-1) - S_{n-1}(-1) + S_{n-2}(-1) = 0, \;\; n \geq 2$$

has the characteristic equation $t^2 - t + 1 = 0$ with roots

$$\cos\frac{\pi}{3} \pm i \sin\frac{\pi}{3}.$$

Therefore

$$S_n(-1) = A\cos\frac{n\pi}{3} + B\sin\frac{n\pi}{3}$$

for all $n \geq 0$ and appropriate constants A and B which can be determined from the initial conditions. Finally we get

$$\sum_{k\geq 0} (-1)^k \binom{n-k}{k} = S_n(-1) = \cos\frac{n\pi}{3} + \frac{1}{\sqrt{3}}\sin\frac{n\pi}{3} = \frac{2}{\sqrt{3}}\cos\frac{(2n-1)\pi}{6}.$$

A period of 6 is immediately detected, based on the periodicity of the trigonometric functions. More precisely $S_{n+6}(-1) = S_n(-1)$ for all $n \geq 0$. The sequence starts $1, 1, 0, -1, -1, 0$, then these values repeat indefinitely. For instance, $S_{6m}(-1) = 1$ for every nonnegative integer m.

H19. *Partition the set of positive integers into $n \geq 1$ arithmetic progressions with first terms $a_1, a_2, \ldots, a_n$ and common differences $d_1, d_2, \ldots, d_n$ respectively. Prove that*

$$\sum_{k=1}^{n} \frac{a_k}{d_k} = \frac{n+1}{2}.$$

Solution. Note that we must have $a_k \leq d_k$. To see this, suppose the contrary $a_k > d_k$. Then $x = a_k - d_k$ must be contained in some arithmetic progression, say the i-th, and we must have $i \neq k$ since x is below the start of the k-th arithmetic progression. But then

$$x + d_i d_k = a_k + (d_i - 1)d_k$$

is in both the i-th and k-th arithmetic progressions, a contradiction. From this it is not hard to see that among the first $N = d_1 d_2 \cdots d_n$ positive integers there are precisely N/d_k numbers belonging to the progression P_k with first term a_k and common difference d_k. This is why we have

$$N = \frac{N}{d_1} + \frac{N}{d_2} + \cdots + \frac{N}{d_n} \Leftrightarrow \sum_{k=1}^{n} \frac{1}{d_k} = 1.$$

Also, we can write

$$\begin{aligned}
\frac{N(N+1)}{2} &= \sum_{j=1}^{N} j = \sum_{k=1}^{n} \sum_{1 \leq j \leq N,\ j \in P_k} j = \sum_{k=1}^{n} \sum_{i=0}^{N/d_k - 1} (a_k + i d_k) \\
&= \sum_{k=1}^{n} \left(\frac{N}{d_k} a_k + \frac{1}{2} \frac{N}{d_k} \left(\frac{N}{d_k} - 1 \right) d_k \right) \\
&= N \sum_{k=1}^{n} \frac{a_k}{d_k} + \frac{N^2}{2} \sum_{k=1}^{n} \frac{1}{d_k} - \frac{Nn}{2} \\
&= N \sum_{k=1}^{n} \frac{a_k}{d_k} + \frac{N^2}{2} - \frac{Nn}{2},
\end{aligned}$$

by using the hypothesis and the observations from the beginning. This can be rearranged as

$$\frac{(n+1)}{2}=\sum_{k=1}^{n}\frac{a_k}{d_k},$$

and the problem is solved.

H20. *Let $a_1 \le a_2 \le \cdots \le a_n$ and $b_1, b_2, \ldots, b_n$ be positive real numbers such that*

$$a_1+a_2+\cdots+a_k \ge b_1+b_2+\cdots+b_k \text{ for all } 1 \le k \le n.$$

Prove that $a_1a_2\cdots a_n \ge b_1b_2\cdots b_n$.

Solution. We have

$$\begin{aligned}\sqrt[n]{a_1a_2\cdots a_n} &\ge \sqrt[n]{a_1a_2\cdots a_n}\frac{1}{n}\left(\frac{b_1}{a_1}+\frac{b_2}{a_2}+\cdots+\frac{b_n}{a_n}\right)\\ &\ge \sqrt[n]{a_1a_2\cdots a_n}\sqrt[n]{\frac{b_1}{a_1}\frac{b_2}{a_2}\cdots\frac{b_n}{a_n}}\\ &= \sqrt[n]{b_1b_2\cdots b_n},\end{aligned}$$

whence the required inequality immediately follows. The second inequality that we used is, of course, the AM-GM inequality, so we still need to explain the first, that is, to prove

$$\frac{b_1}{a_1}+\frac{b_2}{a_2}+\cdots+\frac{b_n}{a_n} \le n.$$

For this one we use Abel's summation formula and the hypothesis:

$$\frac{b_1}{a_1}+\frac{b_2}{a_2}+\cdots+\frac{b_n}{a_n}=\left(\frac{1}{a_1}-\frac{1}{a_2}\right)b_1+\left(\frac{1}{a_2}-\frac{1}{a_3}\right)(b_1+b_2)+\cdots$$

$$+\left(\frac{1}{a_{n-1}}-\frac{1}{a_n}\right)(b_1+\cdots+b_{n-1})+\frac{1}{a_n}(b_1+b_2+\cdots+b_n)$$

$$\leq \left(\frac{1}{a_1} - \frac{1}{a_2}\right) a_1 + \left(\frac{1}{a_2} - \frac{1}{a_3}\right)(a_1 + a_2) + \cdots$$
$$+ \left(\frac{1}{a_{n-1}} - \frac{1}{a_n}\right)(a_1 + \cdots + a_{n-1}) + \frac{1}{a_n}(a_1 + a_2 + \cdots + a_n) = n,$$

and thus we are done.

The problem was proposed by H.A. ShahAli in *Mathematics Magazine* – as Problem 1862 in the February issue from 2011. The above solution, by Omran Kouba, appeared in the February issue of the same *Magazine* from 2012.

H21. *Prove that Carleman's inequality, that is,*

$$\sum_{k=1}^{\infty} \sqrt[k]{a_1 a_2 \cdots a_k} \leq e \sum_{k=1}^{\infty} a_k$$

holds for every positive real numbers $a_1, a_2, \ldots$.

Solution. For positive real numbers $b_1, b_2, \ldots$ we have

$$\sum_{k=1}^{\infty} \sqrt[k]{a_1 a_2 \cdots a_k} = \sum_{k=1}^{\infty} \frac{\sqrt[k]{a_1 b_1 a_2 b_2 \cdots a_k b_k}}{\sqrt[k]{b_1 b_2 \cdots b_k}} \leq \sum_{k=1}^{\infty} \frac{1}{k \sqrt[k]{b_1 b_2 \cdots b_k}} \sum_{j=1}^{k} a_j b_j$$
$$= \sum_{j=1}^{\infty} a_j b_j \sum_{k=j}^{\infty} \frac{1}{k \sqrt[k]{b_1 b_2 \cdots b_k}}$$

by using only the inequality between the arithmetic and geometric means of some positive real numbers, and by changing the order of summation. You can see here a similar trick as in the previous problem in applying the AM-GM inequality, which will be completed by a clever choice of the numbers $b_1, b_2, \ldots$. Namely, we take

$$b_j = \frac{(j+1)^j}{j^{j-1}}$$

for $j = 1, 2, \ldots$, for which

$$b_1 b_2 \cdots b_k = (k+1)^k, \ k = 1, 2, \ldots.$$

We thus get

$$\begin{aligned}\sum_{k=1}^{\infty} \sqrt[k]{a_1 a_2 \cdots a_k} &\le \sum_{j=1}^{\infty} a_j \frac{(j+1)^j}{j^{j-1}} \sum_{k=j}^{\infty} \frac{1}{k(k+1)} \\ &= \sum_{j=1}^{\infty} a_j \frac{(j+1)^j}{j^{j-1}} \frac{1}{j} = \sum_{j=1}^{\infty} \left(1+\frac{1}{j}\right)^j a_j,\end{aligned}$$

which is a little stronger than Carleman's inequality. Indeed, it is well-known that the sequence with general term

$$\left(1+\frac{1}{j}\right)^j, \ j \ge 1$$

is strictly increasing and convergent to the number e, therefore all its terms are less than the limit: we have

$$\left(1+\frac{1}{j}\right)^j < e$$

for every positive integer j. Consequently the inequality follows:

$$\sum_{k=1}^{\infty} \sqrt[k]{a_1 a_2 \cdots a_k} \le \sum_{j=1}^{\infty} \left(1+\frac{1}{j}\right)^j a_j \le e \sum_{j=1}^{\infty} a_j,$$

as desired.

It can be shown that the constant e is optimal (i.e., the smallest possible) in the sense that if, for some C, the inequality

$$\sum_{k=1}^{\infty} \sqrt[k]{a_1 a_2 \cdots a_k} \le C \sum_{k=1}^{\infty} a_k$$

holds for any positive real numbers $a_1, a_2, \ldots$, then $C \ge e$. For example, choose $a_k = 1/k$ for $k = 1, 2, \ldots, n$ and $a_k = 0$ for $k > n$, then let $n \to \infty$ to get this result.

H22. *Prove that*

$$\sum_{k=1}^{n^2} \left\lfloor \sqrt{k} \right\rfloor = \frac{n(4n^2 - 3n + 5)}{6}.$$

Solution. We know that $\left\lfloor \sqrt{k} \right\rfloor = j$ if and only if $j \le \sqrt{k} < j + 1$, that is if and only if $j^2 \le k < (j+1)^2$. Thus, for being able to evaluate the integer parts we write the sum as

$$\begin{aligned}
\sum_{k=1}^{n^2} \left\lfloor \sqrt{k} \right\rfloor &= \sum_{j=1}^{n-1} \sum_{k=j^2}^{(j+1)^2-1} \left\lfloor \sqrt{k} \right\rfloor + \left\lfloor \sqrt{n^2} \right\rfloor = \sum_{j=1}^{n-1} \sum_{k=j^2}^{(j+1)^2-1} j + n \\
&= \sum_{j=1}^{n-1} j(2j+1) + n = 2\sum_{j=1}^{n-1} j^2 + \sum_{j=1}^{n-1} j + n \\
&= \frac{n(n-1)(2n-1)}{3} + \frac{n(n-1)}{2} + n \\
&= \frac{n(4n^2 - 3n + 5)}{6}.
\end{aligned}$$

A question rises naturally: what if the sum is from 1 to some arbitrary number (not necessarily a square)? The answer is that a formula exists. We have

$$\sum_{k=1}^{m} \left\lfloor \sqrt{k} \right\rfloor = (m+1)q - \frac{q(q+1)(2q+1)}{6},$$

for $q = \lfloor \sqrt{m} \rfloor$. The simplest proof goes like this:

$$\begin{aligned}
\sum_{k=1}^{m} \left\lfloor \sqrt{k} \right\rfloor &= \sum_{1 \le k \le m} \sum_{1 \le j \le \sqrt{k}} 1 = \sum_{1 \le j \le \sqrt{m}} \sum_{j^2 \le k \le m} 1 \\
&= \sum_{1 \le j \le \sqrt{m}} (m + 1 - j^2) = \sum_{1 \le j \le q} (m + 1 - j^2) \\
&= (m+1)q - \frac{q(q+1)(2q+1)}{6}.
\end{aligned}$$

The method used in the particular case works in the general case, too. We have (for $q = \lfloor\sqrt{m}\rfloor$, hence for $q^2 \le m < (q+1)^2$)

$$\sum_{k=1}^{m}\left\lfloor\sqrt{k}\right\rfloor = \sum_{j=1}^{q-1}\sum_{k=j^2}^{(j+1)^2-1}\left\lfloor\sqrt{k}\right\rfloor + \sum_{k=q^2}^{m}\left\lfloor\sqrt{k}\right\rfloor$$
$$= \sum_{j=1}^{q-1} j(2j+1) + q(m+1-q^2)$$
$$= (m+1)q - \frac{q(q+1)(2q+1)}{6}$$

after a few simple computations – which we invite you to do. Also, check that the particular formula matches with the general one by replacing $m = n^2$ in the late. Finally, you may wish to prove the (particular or general) formula by induction.

H23. *Let p and q be relatively prime odd natural numbers. Prove that*

$$\sum_{k=1}^{\frac{p-1}{2}}\left\lfloor\frac{kq}{p}\right\rfloor + \sum_{l=1}^{\frac{q-1}{2}}\left\lfloor\frac{lp}{q}\right\rfloor = \frac{(p-1)(q-1)}{4}.$$

Solution. We have

$$\sum_{k=1}^{\frac{p-1}{2}}\left\lfloor\frac{kq}{p}\right\rfloor = \sum_{k=1}^{\frac{p-1}{2}}\sum_{l=1}^{\left\lfloor\frac{kq}{p}\right\rfloor} 1 = \sum_{1\le k\le\frac{p-1}{2}}\ \sum_{1\le l\le\left\lfloor\frac{kq}{p}\right\rfloor} 1 = \sum_{1\le k<\frac{p}{2}}\ \sum_{1\le l<\frac{kq}{p}} 1$$

$$= \sum_{1\le l<\frac{q}{2}}\ \sum_{\frac{lp}{q}<k<\frac{p}{2}} 1 = \sum_{1\le l\le\frac{q-1}{2}}\ \sum_{\left\lfloor\frac{lp}{q}\right\rfloor+1\le k\le\frac{p-1}{2}} 1$$

$$= \sum_{l=1}^{\frac{q-1}{2}}\left(\sum_{k=1}^{\frac{p-1}{2}} 1 - \sum_{k=1}^{\left\lfloor\frac{lp}{q}\right\rfloor} 1\right) = \sum_{l=1}^{\frac{q-1}{2}}\sum_{k=1}^{\frac{p-1}{2}} 1 - \sum_{l=1}^{\frac{q-1}{2}}\sum_{k=1}^{\left\lfloor\frac{lp}{q}\right\rfloor} 1$$

$$= \frac{p-1}{2} \cdot \frac{q-1}{2} - \sum_{l=1}^{\frac{q-1}{2}} \left\lfloor \frac{lp}{q} \right\rfloor,$$

as desired. We used the fact that, for natural numbers k and l we have $k < p/2$ if and only if $k \leq (p-1)/2$, and $l < q/2$ if and only if $l \leq (q-1)/2$ (because p and q are odd). We also used the fact that no kq/p with $1 \leq k \leq (p-1)/2$, and no lp/q with $1 \leq l \leq (q-1)/2$ can be an integer, as p and q are relatively prime (actually this holds for $1 \leq k < p$ and $1 \leq l < q$ respectively).

One can figure out a geometric proof, too. Namely, consider in the xy-plane (with origin O) the points $P(p/2, 0)$, $Q(0, q/2)$ and $R(p/2, q/2)$. It is easy to see that the number of points with integer coordinates (lattice points) that lie inside the rectangle $OPRQ$ is $(p-1)(q-1)/4$. On the other hand, there are no lattice points on the diagonal OR of the rectangle, there are

$$\sum_{k=1}^{\frac{p-1}{2}} \left\lfloor \frac{kq}{p} \right\rfloor$$

lattice points inside the triangle OPR, and there are

$$\sum_{l=1}^{\frac{q-1}{2}} \left\lfloor \frac{lp}{q} \right\rfloor$$

lattice points inside the triangle OQR. The conclusion follows by equating the two expressions of the number of lattice points situated inside the rectangle $OPRQ$. Of course, we invite the reader to prove all these statements. Also, we invite you to prove (by using the first method) that the more general identity

$$\sum_{k=1}^{r} \left\lfloor \frac{kq}{p} \right\rfloor + \sum_{l=1}^{\left\lfloor \frac{rq}{p} \right\rfloor} \left\lfloor \frac{lp}{q} \right\rfloor = r \left\lfloor \frac{rq}{p} \right\rfloor$$

holds for *any* positive and relatively prime integers p and q (not necessarily both odd), and any $1 \leq r < p$.

H24. *Let p be an odd prime. Prove that*

$$\sum_{k=1}^{p-1}\left\lfloor\frac{k^3}{p}\right\rfloor=\frac{(p-2)(p-1)(p+1)}{4}.$$

Solution. We have

$$\sum_{k=1}^{p-1}\left\lfloor\frac{k^3}{p}\right\rfloor=\sum_{k=1}^{(p-1)/2}\left(\left\lfloor\frac{k^3}{p}\right\rfloor+\left\lfloor\frac{(p-k)^3}{p}\right\rfloor\right)$$

by pairing the first term with the last, the second with the last but one, and so on. But

$$\frac{(p-k)^3}{p}=p^2-3pk+3k^2-\frac{k^3}{p}$$

and if we use $\lfloor x+n\rfloor=\lfloor x\rfloor+n$ (for any real number x and any integer n) and $\lfloor x\rfloor+\lfloor -x\rfloor=-1$ (for any x which is not an integer) we have

$$\left\lfloor\frac{k^3}{p}\right\rfloor+\left\lfloor\frac{(p-k)^3}{p}\right\rfloor=p^2-3pk+3k^2-1$$

for every $k=1,2,\ldots,(p-1)/2$. Thus our sum becomes

$$\begin{aligned}\sum_{k=1}^{p-1}\left\lfloor\frac{k^3}{p}\right\rfloor&=p^2\frac{p-1}{2}-3p\sum_{k=1}^{(p-1)/2}k+3\sum_{k=1}^{(p-1)/2}k^2-\frac{p-1}{2}\\&=\frac{p^2(p-1)}{2}-\frac{3p(p-1)(p+1)}{8}+\frac{p(p-1)(p+1)}{8}-\frac{p-1}{2}\\&=\frac{(p-2)(p-1)(p+1)}{4}.\end{aligned}$$

H25. *Let p be an odd prime and let $f:\mathbb{Z}_+\to\mathbb{R}$ be a function such that*

(i) $\dfrac{f(k)}{p}$ *is not an integer, for $k=1,2,\ldots,p-1$;*

(ii) $f(k)+f(p-k)$ is an integer divisible by p, for $k=1,2,\ldots,p-1$. Prove that

$$\sum_{k=1}^{p-1}\left\lfloor\frac{f(k)}{p}\right\rfloor=\frac{1}{p}\sum_{k=1}^{p-1}f(k)-\frac{p-1}{2}.$$

Solution. From (ii) it follows that $\dfrac{f(k)}{p}+\dfrac{f(p-k)}{p}\in\mathbb{Z}$ and therefore

$$\left\{\frac{f(k)}{p}\right\}+\left\{\frac{f(p-k)}{p}\right\}\in\mathbb{Z}.$$

From (i) we obtain that $\dfrac{f(k)}{p}\notin\mathbb{Z}$ and $\dfrac{f(p-k)}{p}\notin\mathbb{Z}$, $k=1,2,\ldots,p-1$. Thus

$$0<\left\{\frac{f(k)}{p}\right\}+\left\{\frac{f(p-k)}{p}\right\}<2.$$

Using the above results we get

$$\left\{\frac{f(k)}{p}\right\}+\left\{\frac{f(p-k)}{p}\right\}=1,\ \text{for } k=1,2,\ldots,p-1.$$

Summing up yields

$$\sum_{k=1}^{p-1}\left\{\frac{f(k)}{p}\right\}=\frac{1}{2}\sum_{k=1}^{p-1}\left(\left\{\frac{f(k)}{p}\right\}+\left\{\frac{f(p-k)}{p}\right\}\right)=\frac{p-1}{2}.$$

It follows that

$$\sum_{k=1}^{p-1}\left\lfloor\frac{f(k)}{p}\right\rfloor=\frac{1}{p}\sum_{k=1}^{p-1}f(k)-\frac{p-1}{2}.$$

The careful reader definitely recognized here a generalization of the previous problem (whose result follows from this one by considering $f(n)=n^3$).

H26. *If $p>3$ is a prime number and x, y, and z are integers such that $x+y+z$ and $xy+xz+yz$ are both divisible by p, then $x^p+y^p+z^p$ and $x^py^p+x^pz^p+y^pz^p$ are divisible by p^2.*

Solution 1. First we observe that

$$(t+1)^p-(t^p+1)=p(t^2+t+1)Q(t),$$

where Q is a polynomial with integer coefficients.

Indeed, we know that $P(t) = (t+1)^p - (t^p+1)$ has all coefficients divisible by p, since they are the binomial coefficients $\binom{p}{j}$ for $j = 1, 2, \ldots, p-1$. Moreover, if α is a zero of $t^2 + t + 1$ (thus a third root of unity different from 1), we have $\alpha^2 + \alpha + 1 = 0$ and $\alpha^3 = 1$, consequently

$$P(\alpha) = (\alpha+1)^p - (\alpha^p + 1) = (-\alpha^2)^p - \alpha^p - 1 = -(\alpha^2 + \alpha + 1) = 0.$$

(If $p \equiv 1 \pmod 3$, we have $\alpha^{2p} = \alpha^2$ and $\alpha^p = \alpha$; if $p \equiv 2 \pmod 3$, we have $\alpha^{2p} = \alpha$ and $\alpha^p = \alpha^2$.) We infer that $P(t)/p$ is a polynomial with integer coefficients divisible by $t^2 + t + 1$, whence the conclusion follows.

By using the above with x/y in place of t we get

$$(x+y)^p - (x^p + y^p) = p(x^2 + xy + y^2)R(x, y),$$

where $R(x, y)$ is a (homogeneous) polynomial with integer coefficients.

Now we solve the problem. Because

$$x + y \equiv -z \pmod p \quad \text{and} \quad xy + xz + yz \equiv 0 \pmod p,$$

we get

$$\begin{aligned} x^2 + xy + y^2 &= (x+y)^2 - xy \equiv -z(x+y) - xy \\ &= -(xy + xz + yz) \equiv 0 \pmod p, \end{aligned}$$

hence the above equality shows that p^2 divides $(x+y)^p - (x^p + y^p)$. But the same is true for $(x+y)^p + z^p$, because

$$(x+y)^p + z^p = (x+y+z)\left(\sum_{j=0}^{p-1} (x+y)^{p-1-j}(-z)^j\right),$$

and both factors from the right hand side are divisible by p. Indeed, we have $-z \equiv x + y \pmod p$, thus

$$\begin{aligned} \sum_{j=0}^{p-1} (x+y)^{p-1-j}(-z)^j &\equiv \sum_{j=0}^{p-1} (x+y)^{p-1-j}(x+y)^j \\ &= p(x+y)^{p-1} \equiv 0 \pmod p. \end{aligned}$$

Consequently

$$x^p + y^p + z^p = ((x+y)^p + z^p) - ((x+y)^p - (x^p + y^p))$$

is divisible by p^2, as claimed.

On the other hand, note that the numbers xy, xz, and yz also fulfill the hypotheses: their sum $xy + xz + yz$ and the sum of their two by two products

$$xyxz + xyyz + xzyz = xyz(x+y+z)$$

are also divisible by p. By what we already proved, $(xy)^p + (xz)^p + (yz)^p$ is divisible by p^2, too, and thus we are done.

Solution 2. Let $P_k = x^k + y^k + z^k$. Since x, y, z are the three roots of the polynomial

$$(X-x)(X-y)(X-z) = X^3 - (x+y+z)X^2 + (xy+yz+zx)X - xyz,$$

we have

$$x^k - (x+y+z)x^{k-1} + (xy+yz+zx)x^{k-2} - xyzx^{k-3} = 0$$

for $k \geq 3$ and similarly for y and z. Thus the P_k satisfy the recursion

$$P_0 = 3, \quad P_1 = x+y+z, \quad P_2 = (x+y+z)^2 - 2(xy+yz+zx),$$

and

$$P_k = (x+y+z)P_{k-1} - (xy+yz+zx)P_{k-2} + xyzP_{k-3}$$

for $k \geq 3$. From this recursion and the hypotheses an easy induction shows that

$$P_k \equiv \begin{cases} k(x+y+z)(xyz)^{(k-1)/2} \pmod{p^2} & \text{if } k \equiv 1 \pmod 3 \\ -k(xy+yz+zx)(xyz)^{(k-2)/2} \pmod{p^2} & \text{if } k \equiv 2 \pmod 3 \\ 3(xyz)^{k/3} \pmod{p^2} & \text{if } k \equiv 0 \pmod 3 \end{cases}.$$

In more detail, if $k = 3m$, then by the inductive hypothesis we see that p divides P_{3m-1} and P_{3m-2} so p^2 divides $(x+y+z)P_{3m-1}$ and $(xy+yz+zx)P_{3m-2}$. Thus the recursion gives

$$P_{3m} \equiv xyzP_{3m-3} \equiv xyz \cdot 3(xyz)^{m-1} = 3(xyz)^m \pmod{p^2}.$$

If $k = 3m+1$, then p^2 divides $(xy+yz+zx)P_{3m-1}$ and the recursion gives

$$\begin{aligned} P_{3m+1} &\equiv (x+y+z)P_{3m} + xyzP_{3m-2} \\ &\equiv 3(x+y+z)(xyz)^m + (3m-2)(x+y+z)(xyz)^m \\ &= (3m+1)(x+y+z)(xyz)^m \pmod{p^2}. \end{aligned}$$

The case $k = 3m+2$ is similar. In particular, since $p > 3$, we find that P_p is in one of the first two cases and hence in either case $P_p \equiv 0 \pmod{p^2}$. The conclusion about the divisibility of the second sum with p^2 follows as in the first solution.

H27. *Let p be an odd prime and let*

$$S_q = \frac{1}{2\cdot 3\cdot 4} + \frac{1}{5\cdot 6\cdot 7} + \cdots + \frac{1}{q(q+1)(q+2)},$$

where $q = \dfrac{3p-5}{2}$. Assume that $\dfrac{1}{p} - 2S_q = \dfrac{m}{n}$, for some integers m and n. Prove that $m \equiv n (\mathrm{mod}\ p)$.

Solution. Let $p = 2s+1$, so that we have $q = 3s-1$, and the sum becomes

$$\begin{aligned} S_q &= \frac{1}{2\cdot 3\cdot 4} + \frac{1}{5\cdot 6\cdot 7} + \cdots + \frac{1}{q(q+1)(q+2)} \\ &= \sum_{k=1}^{s} \frac{1}{(3k-1)(3k)(3k+1)} \\ &= \sum_{k=1}^{s} \frac{1}{2}\left(\frac{1}{3k-1} + \frac{1}{3k} + \frac{1}{3k+1} - \frac{1}{k}\right) \\ &= \frac{1}{2}\left(\sum_{k=2}^{3s+1}\frac{1}{k} - \sum_{k=1}^{s}\frac{1}{k}\right) = \frac{1}{2}\left(-1 + \sum_{k=s+1}^{3s+1}\frac{1}{k}\right). \end{aligned}$$

Consequently,

$$\frac{m}{n} = \frac{1}{p} - 2S_q = \frac{1}{2s+1} + 1 - \sum_{k=s+1}^{3s+1} \frac{1}{k}$$

and

$$\frac{n-m}{n} = \sum_{j=1}^{s} \left(\frac{1}{2s+1-j} + \frac{1}{2s+1+j} \right) = \sum_{j=1}^{s} \frac{4s+2}{(2s+1-j)(2s+1+j)}$$
$$= 2\sum_{j=1}^{s} \frac{p}{(p-j)(p+j)}.$$

Clearly, after getting the same denominator to the last sum, the factor p remains in the numerator (it cannot be cancelled, because the denominator $\prod_{j=1}^{s}(p-j)(p+j)$ does not have a factor of p), meaning that p is a divisor of $n-m$, that is, $m \equiv n \pmod{p}$.

H28. *Let n be a positive integer, and let 2^r be the highest power of 2 dividing n. Prove that 2^{2r} is the highest power of 2 dividing the numerator of*

$$1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2n-1}$$

when the sum is represented as a fraction in its lowest terms.

Solution. Write $n = 2^r m$ where m is odd. Then the given sum can be split into a sum (over $s = 0, \ldots, m-1$) of m sums of the form

$$S_s = \sum_{j=1}^{2^r} \frac{1}{2^{r+1}s + 2j - 1}.$$

We will first show that if we write such a sum S_s as a fraction in lowest terms, then $S_s = 2^{2r}a/b$ where a and b are odd. Replacing j by 2^r+1-j in the sum above (that is, taking the terms in reverse order), we find

$$S_s = \sum_{j=1}^{2^r} \frac{1}{2^{r+1}(s+1) - 2j + 1}.$$

Averaging these two expressions we get

$$S_s = \frac{1}{2}\sum_{j=1}^{2^r}\left(\frac{1}{2^{r+1}s+2j-1}+\frac{1}{2^{r+1}(s+1)-2j+1}\right)$$
$$= \sum_{j=1}^{2^r}\frac{2^r(2s+1)}{(2^{r+1}s+2j-1)(2^{r+1}(s+1)-2j+1)}.$$

Factoring out $2^r(2s+1)$, we see that it is enough to show that if we write

$$T_s = \sum_{j=1}^{2^r}\frac{1}{(2^{r+1}s+2j-1)(2^{r+1}(s+1)-2j+1)}$$

as a fraction in lowest terms, then $T_s = 2^r a'/b'$ where a' and b' are odd. For each $j = 1, \dots, 2^r$, there is a k_j with $1 \le k_j \le 2^r$ such that

$$(2j-1)(2k_j-1) \equiv 1 \pmod{2^{r+1}}.$$

Further as j varies $2j-1$ runs over all the distinct odd congruence classes modulo 2^{r+1}, hence the $2k_j-1$ also runs over all distinct odd congruence classes modulo 2^{r+1}. Hence the numbers k_j are the numbers $1, 2, \dots, 2^r$ in some order. The equation

$$(2j-1)(2k_j-1) \equiv 1 \pmod{2^{r+1}}$$

implies that if we write

$$\frac{1}{(2^{r+1}s+2j-1)(2^{r+1}(s+1)-2j+1)} + (2k_j-1)^2$$

as a fraction in lowest terms, then the numerator will be a multiple of 2^{r+1}. Hence the same will be true of their sum, which is

$$T_s + \sum_{k=1}^{2^r}(2k-1)^2 = T_s + \frac{2^r(2^{2r+2}-1)}{3}.$$

Thus we see that $T_s = 2^r a'/b'$ where a' and b' are odd, as desired. Since this means each $S_s = 2^{2r} a/b$ with a and b odd, and since we are adding an odd number m of such terms, the sum also has the form $2^{2r} a/b$ with a and b odd.

This was proposed as problem E1408 in *The American Mathematical Monthly* by J.L. Selfridge.

H29. *Let $n \geq 2$ be a positive integer, with divisors $1 = d_1 < d_2 < \cdots < d_k = n$. Prove that $d_1d_2 + d_2d_3 + \cdots + d_{k-1}d_k$ is always less than n^2, and determine when it is a divisor of n^2.*

Solution. If $d_1, d_2, \ldots, d_k$ are all the positive divisors of n, then n/d_1, $n/d_2, \ldots, n/d_k$ are the same divisors (in reverse order).
Consequently,

$$S = d_1d_2 + d_2d_3 + \cdots + d_{k-1}d_k = n^2\left(\frac{1}{d_1d_2} + \frac{1}{d_2d_3} + \cdots + \frac{1}{d_{k-1}d_k}\right)$$

$$= n^2\left(\left(\frac{1}{d_1} - \frac{1}{d_2}\right)\frac{1}{d_2 - d_1} + \left(\frac{1}{d_2} - \frac{1}{d_3}\right)\frac{1}{d_3 - d_2} + \cdots\right.$$

$$\left. + \left(\frac{1}{d_{k-1}} - \frac{1}{d_k}\right)\frac{1}{d_k - d_{k-1}}\right)$$

$$\leq n^2\left(\left(\frac{1}{d_1} - \frac{1}{d_2}\right) + \left(\frac{1}{d_2} - \frac{1}{d_3}\right) + \cdots + \left(\frac{1}{d_{k-1}} - \frac{1}{d_k}\right)\right)$$

$$= n^2\left(\frac{1}{d_1} - \frac{1}{d_k}\right) < \frac{n^2}{d_1} = n^2,$$

because each difference $d_i - d_{i-1}$ is at least 1. Or, alternatively, we can use the fact that, for positive integers $d_1 < d_2 < \cdots < d_k$ (not necessarily the divisors of some number) we have $d_i \geq i$ – which is easy to prove by induction, based on the same idea that if a and b are positive integers

with $a < b$, then $a + 1 \le b$. So, we have

$$\begin{aligned} S &= d_1d_2 + d_2d_3 + \cdots + d_{k-1}d_k \\ &= n^2\left(\frac{1}{d_1d_2} + \frac{1}{d_2d_3} + \cdots + \frac{1}{d_{k-1}d_k}\right) \\ &\le n^2\left(\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{(k-1)\cdot k}\right) \\ &= n^2\left(1 - \frac{1}{k}\right) < n^2. \end{aligned}$$

For the second part, let us suppose that S divides n^2, and let p be the least prime divisor of n. (Thus, actually, $d_2 = p$.) If we have $k = 2$, then $n = p$ (is a prime) and $S = p = n$ divides $p^2 = n^2$. So, the prime numbers are solutions to our problem.

Otherwise, $k \ge 3$ and S is strictly greater than $d_{k-1}d_k = n^2/p$. It follows that p is also strictly greater than n^2/S, which, by the first part of the problem, is greater than 1. However, we want S to be a divisor of n^2, meaning that n^2/S is also a divisor of n^2. Thus we obtained a divisor of n^2 between 1 and p, its smallest prime divisor, which is impossible (hence that's impossible for S to divide n^2 in this case).

It follows that S is a divisor of n^2 if and only if n is a prime. Thus the prime numbers are all the solutions to our problem – which was one of the questions asked in the International Mathematical Olympiad in the year 2002.

H30. *Prove that*

$$\sum_{d|n} \frac{\mu(d)}{d} = \frac{\phi(n)}{n}.$$

Solution. Remember that for an arithmetic function f and its summation function F defined by

$$F(n) = \sum_{d|n} f(d)$$

for all $n \geq 1$, we also have, by Möbius inversion formula,

$$f(n) = \sum_{d|n} \mu(d) F\left(\frac{n}{d}\right)$$

for every positive integer n. Also, for f being Euler's function ϕ, we have

$$F(n) = \sum_{d|n} \phi(d) = n$$

for every $n \geq 1$. Therefore, by Möbius inversion,

$$\phi(n) = \sum_{d|n} \mu(d) \frac{n}{d},$$

which gives the desired result after dividing by n.

H31. *Prove that*

$$\sum_{d|n} \sigma(d) \mu\left(\frac{n}{d}\right) = n.$$

Solution. This is Möbius inversion again. The function σ is defined by

$$\sigma(n) = \sum_{d|n} d$$

(the sum of divisors of n) for any positive integer n. That is, σ is nothing else but the summation function of the function $f(n) = n$. Thus, by Möbius inversion,

$$n = \sum_{d|n} \mu(d) \sigma\left(\frac{n}{d}\right) = \sum_{d|n} \mu\left(\frac{n}{d}\right) \sigma(d).$$

The last equality holds because when d runs over all positive divisors of n, n/d does exactly the same thing (in reverse order).

H32. *Prove that*

$$\prod_{d|n} d^{\mu(d)} = \begin{cases} 1, & \text{if } n \text{ is not a power of a prime} \\ \dfrac{1}{p}, & \text{if } n = p^a, \text{ with } p \text{ prime.} \end{cases}$$

Solution. For $n = 1$ the result is clear. For $n = p^a$ we have, indeed,

$$\prod_{d|p^a} d^{\mu(d)} = 1^{\mu(1)} p^{\mu(p)} = p^{-1}.$$

For $n = p_1^{a_1} \cdots p_s^{a_s}$, with prime $p_1, \ldots, p_s$ and $s \geq 2$, all the factors of the products corresponding to divisors $d = p_1^{b_1} \cdots p_s^{b_s}$ with at least one $b_i \geq 2$ are 1, therefore the product becomes

$$\prod_{d|n} d^{\mu(d)} = p_1^{-1} \cdots p_s^{-1} (p_1 p_2) \cdots (p_{s-1} p_s)(p_1 p_2 p_3)^{-1} \cdots$$
$$\cdot (p_{s-2} p_{s-1} p_s)^{-1} \cdots (p_1 p_2 \cdots p_s)^{(-1)^s}.$$

(When we say $(p_1 p_2) \cdots (p_{s-1} p_s)$ we understand the product of all products of two distinct factors chosen from $p_1, \ldots, p_s$, and so on.) Since each p_i occurs as a factor of $\binom{s-1}{k-1}$ products of k distinct factors, we see that the exponent of each p_i is

$$-\binom{s-1}{0} + \binom{s-1}{1} - \binom{s-1}{2} + \cdots + (-1)^s \binom{s-1}{s-1} = 0$$

(and it is important here that $s - 1 \geq 1$), we get the product equal to 1, as required to prove.

H33. *Let a_n be a sequence of integers that satisfies*

$$\sum_{d|n} a_d = 2^n \text{ for all } n \geq 1.$$

Prove that $n \mid a_n$ for all $n \geq 1$.

Solution. By Möbius inversion formula we have

$$a_n = \sum_{d|n} \mu(d) 2^{\frac{n}{d}}.$$

If $n = p_1^{r_1} \cdots p_s^{r_s}$ is the factorization of the (arbitrary) positive integer n (with $p_1, \ldots, p_s$ prime numbers, and $r_1, \ldots, r_s$ positive integers), for proving $n | a_n$ it suffices to show that $p_i^{r_i}$ divides a_n for every $1 \leq i \leq n$. Let p^r be any of the factors $p_i^{r_i}$ (we give up the index for the sake of simplicity of notation). Then we have $n = p^r m$, with $(m, p) = 1$, and, by the definition of the Möbius function, the expression of a_n becomes

$$\begin{aligned} a_n &= \sum_{d|m} \mu(d) 2^{\frac{n}{d}} + \sum_{d|m,\ 1 \leq b \leq r} \mu(p^b d) 2^{\frac{n}{p^b d}} = \sum_{d|m} \mu(d) 2^{\frac{n}{d}} + \sum_{d|m} \mu(pd) 2^{\frac{n}{pd}} \\ &= \sum_{d|m} \mu(d) \left(2^{\frac{n}{d}} - 2^{\frac{n}{pd}} \right) = \sum_{d|m} \mu(d) \left(2^{p^r \frac{m}{d}} - 2^{p^{r-1} \frac{m}{d}} \right) \\ &= \sum_{d|m} \mu(d) 2^{p^{r-1} \frac{m}{d}} \left(2^{(p^r - p^{r-1}) \frac{m}{d}} - 1 \right). \end{aligned}$$

If we prove that

$$2^{p^{r-1} \frac{m}{d}} \left(2^{(p^r - p^{r-1}) \frac{m}{d}} - 1 \right)$$

is divisible by p^r, the divisibility of a_n by p^r will follow, and the problem will be solved. This is clear when $p = 2$, because $2^{r-1} \geq r$ for positive integer r and the first factor $2^{2^{r-1} m/d}$ is therefore divisible by 2^r. For odd prime p, by Euler's theorem we have

$$2^{p^r - p^{r-1}} = 2^{\phi(p^r)} \equiv 1 \pmod{p^r}.$$

Raising this congruence to the m/dth power gives

$$2^{(p^r - p^{r-1}) \frac{m}{d}} \equiv 1 \pmod{p^r},$$

hence

$$2^{p^{r-1} \frac{m}{d}} \left(2^{(p^r - p^{r-1}) \frac{m}{d}} - 1 \right)$$

is divisible by p^r (because the expression in parentheses is), as we intended to prove. The solution ends here.

The problem was shortlisted for the International Mathematical Olympiad in 1989.

H34. *Prove that*

$$\sum_{n=1}^{\infty} \frac{\phi(n)}{2^n - 1} = 2.$$

Solution. By the summation formula for a geometric series

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots, \ |x| < 1,$$

we have

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{\phi(n)}{2^n - 1} &= \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n \frac{\phi(n)}{1 - \left(\frac{1}{2}\right)^n} = \sum_{n=1}^{\infty} \phi(n) \left(\frac{1}{2}\right)^n \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^{kn} \\
&= \sum_{n=1}^{\infty} \phi(n) \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^{(k+1)n} = \sum_{n=1}^{\infty} \sum_{l=1}^{\infty} \phi(n) \left(\frac{1}{2}\right)^{ln} \\
&= \sum_{m=1}^{\infty} \left(\sum_{d|m} \phi(d)\right) \left(\frac{1}{2}\right)^m = \sum_{m\geq 1} \frac{m}{2^m} = 2.
\end{aligned}$$

H35. *Let p be a positive prime, and let r be a positive integer. Consider the positive integers n and m such that $n \geq m > p^r - p^{r-1}$ and the integers $a_1, \ldots, a_n$. For any $0 \leq j \leq n$ denote by s_j and t_j the number of sums of the form $a_{i_1} + \cdots + a_{i_j}$, with $1 \leq i_1 < \cdots < i_j \leq n$ which are, and, respectively, which are not divisible by p (thus $s_0 = 1$, $t_0 = 0$). Prove that*

$$S = \sum_{j=0}^{m} (-1)^j \binom{n-m+j}{j} s_{m-j} \equiv 0 \pmod{p^r}$$

and

$$T = \sum_{j=0}^{m} (-1)^j \binom{n-m+j}{j} t_{m-j} \equiv 0 \pmod{p^r}.$$

Solution. We start with the identity

$$(x_1+\cdots+x_m)^k - \sum_{1\le i_1<\cdots<i_{m-1}\le m}(x_{i_1}+\cdots+x_{i_{m-1}})^k$$

$$+\sum_{1\le i_1<\cdots<i_{m-2}\le m}(x_{i_1}+\cdots+x_{i_{m-2}})^k - \cdots + (-1)^{m-1}\sum_{1\le i_1\le m} x_{i_1}^k = 0$$

for $k<m$ and for any complex numbers $x_1,\ldots,x_m$ (see Example 5.6 in the chapter *Mathematical Induction*). We replace here $(x_1,\ldots,x_m)$ by all possible m-tuples selected from the given integer numbers $a_1,\ldots,a_n$ and sum all the equalities of this type in order to get

$$\sum_{1\le i_1<\cdots<i_m\le n}(a_{i_1}+\cdots+a_{i_m})^k$$

$$-\binom{n-m+1}{1}\sum_{1\le i_1<\cdots<i_{m-1}\le n}(a_{i_1}+\cdots+a_{i_{m-1}})^k$$

$$+\binom{n-m+2}{2}\sum_{1\le i_1<\cdots<i_{m-2}\le n}(a_{i_1}+\cdots+a_{i_{m-2}})^k - \cdots$$

$$+(-1)^{m-1}\binom{n-1}{m-1}\sum_{1\le i_1\le m} a_{i_1}^k = 0.$$

We can write this in the more condensed form

$$\sum_{j=0}^{m-1}(-1)^j\binom{n-m+j}{j}\sum_{1\le i_1<\cdots<i_{m-j}\le n}(a_{i_1}+\cdots+a_{i_{m-j}})^k = 0.$$

We choose here $k=\phi(r)=p^r-p^{r-1}<m$, hence (by Fermat's little theorem) every term $(a_{i_1}+\cdots+a_{i_{m-j}})^k$ of the above sums is congruent either to 0, or to 1 modulo p^r, depending on whether the sum $a_{i_1}+\cdots+a_{i_{m-j}}$ is, or is not, divisible by p. (When p divides an integer x, p^r

divides x^k, since $k = p^r - p^{r-1} = p^{r-1}(p-1) \geq 2^{r-1}(2-1) = 2^{r-1} \geq r$.) Consequently, considering the last equality modulo p^r yields

$$\sum_{j=0}^{m-1} (-1)^j \binom{n-m+j}{j} t_{m-j} \equiv 0 \pmod{p^r}.$$

But this is the same as

$$T = \sum_{j=0}^{m} (-1)^j \binom{n-m+j}{j} t_{m-j} \equiv 0 \pmod{p^r}$$

because $t_0 = 0$.

For the first congruence we still have to see that, obviously, $s_j + t_j = \binom{n}{j}$ for every $0 \leq j \leq n$, hence

$$\begin{aligned} S + T &= \sum_{j=0}^{m} (-1)^j \binom{n-m+j}{j} \binom{n}{m-j} \\ &= \binom{n}{m} \sum_{j=0}^{m} (-1)^j \binom{m}{j} = 0; \end{aligned}$$

thus, $T \equiv 0 \pmod{p}$ implies $S \equiv 0 \pmod{p}$, too.

Note that the congruences still hold for $m = p^r - p^{r-1}$ if $(p^r - p^{r-1})!$ is divisible by p^r (that is, if $(r, m) \neq (1, p-1)$ and $(p, r, m) \neq (2, 2, 2)$, as one can see). In order to prove this it is enough to use the identity

$$\sum_{j=0}^{m-1} (-1)^j \sum_{1 \leq i_1 < \cdots < i_{m-j} \leq m} (x_{i_1} + \cdots + x_{i_{m-j}})^m = m! x_1 \cdots x_m$$

(again see Example 5.6 in the chapter *Mathematical Induction*), then proceed similarly to the proof above. The complete statement (with the case $m = p^r - p^{r-1}$ included) is problem 11391 from *The American Mathematical Monthly*; Richard Stong gave a solution and a strong generalization in the issue from December 2010 of the same magazine.

H36. *Evaluate*

$$\frac{11}{1\cdot 2\cdot 3}\left(\frac{4}{5}\right)+\frac{12}{2\cdot 3\cdot 4}\left(\frac{4}{5}\right)^2+\frac{13}{3\cdot 4\cdot 5}\left(\frac{4}{5}\right)^3+\cdots$$

Solution. Although it is not so evident, we have a possibility of telescoping this sum. Namely,

$$\sum_{n=1}^{\infty}\frac{n+10}{n(n+1)(n+2)}\left(\frac{4}{5}\right)^n$$

$$=\sum_{n=1}^{\infty}5\left(\frac{1}{n(n+1)}\left(\frac{4}{5}\right)^n-\frac{1}{(n+1)(n+2)}\left(\frac{4}{5}\right)^{n+1}\right)=5\cdot\frac{1}{1\cdot 2}\cdot\frac{4}{5}=2.$$

Indeed, in view of the (for us now well-known) formula

$$\frac{1}{n(n+1)(n+2)}=\frac{1}{2}\left(\frac{1}{n(n+1)}-\frac{1}{(n+1)(n+2)}\right),$$

we are entitled to assume that something like

$$\frac{n+10}{n(n+1)(n+2)}\left(\frac{4}{5}\right)^n=a\left(\frac{1}{n(n+1)}\left(\frac{4}{5}\right)^n-\frac{1}{(n+1)(n+2)}\left(\frac{4}{5}\right)^{n+1}\right),$$

would hold for some real number a. And it holds for $a=5$, doesn't it?

Note that the following formulae hold for $x\in(-1,1)$:

$$\sum_{n=1}^{\infty}\frac{x^n}{n}=-\ln(1-x),$$

$$\sum_{n=1}^{\infty}\frac{x^n}{n(n+1)}=1+\frac{1-x}{x}\ln(1-x),$$

and

$$\sum_{n=1}^{\infty}\frac{x^n}{n(n+1)(n+2)}=\frac{3}{4}-\frac{1}{2x}-\frac{1}{2}\left(\frac{1-x}{x}\right)^2\ln(1-x).$$

(You can integrate twice with respect to x the first equality – which we saw in Chapter 6 – in order to obtain the second and the third.) If you properly combine the second and the third of these formulae, you obtain another solution to our problem.

Nevertheless, with or without these identities, we are sure that the reader is now able to compute

$$\sum_{n=1}^{\infty} \frac{2+(1-x)n}{n(n+1)(n+2)} x^n$$

for some given real number $x \in (-1, 1)$.

H37. *Let a be a real number. Define the sequence $(x_n)_{n\geq 1}$ recursively by*

$$x_1 = 1 \textit{ and } x_{n+1} = a^n + nx_n, \textit{ for } n \geq 1.$$

Prove that

$$\prod_{n=1}^{\infty} \left(1 - \frac{a^n}{x_{n+1}}\right) = e^{-a}.$$

Solution. We can rewrite the recurrence relation as

$$\frac{x_{n+1}}{n!} = \frac{a^n}{n!} + \frac{x_n}{(n-1)!},$$

and we easily iterate this to yield

$$\frac{x_{n+1}}{n!} = 1 + \frac{a}{1!} + \frac{a^2}{2!} + \cdots + \frac{a^n}{n!},$$

which holds for all $n \geq 0$. Thus $x_{n+1} = n! e_n(a)$, where

$$e_n(a) = 1 + \frac{a}{1!} + \frac{a^2}{2!} + \cdots + \frac{a^n}{n!}$$

tends to e^a as $n \to \infty$, as we know (basically this is the definition of the exponential with power series). Now

$$1 - \frac{a^n}{x_{n+1}} = 1 - \frac{\frac{a^n}{n!}}{e_n(a)} = \frac{e_{n-1}(a)}{e_n(a)},$$

therefore

$$\prod_{n=1}^{N}\left(1-\frac{a^n}{x_{n+1}}\right)=\prod_{n=1}^{N}\frac{e_{n-1}(a)}{e_n(a)}=\frac{e_0(a)}{e_N(a)}=\frac{1}{e_N(a)}$$

for each positive integer N. Finally

$$\prod_{n=1}^{N}\left(1-\frac{a^n}{x_{n+1}}\right)=\lim_{N\to\infty}\prod_{n=1}^{N}\left(1-\frac{a^n}{x_{n+1}}\right)=\lim_{N\to\infty}\frac{1}{e_N(a)}=\frac{1}{e^a},$$

as required.

H38. *Evaluate*

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{i!j!}{(i+j+2)!}.$$

Solution. In the solution to problem M36 we calculated

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{i!j!}{(i+j+1)!}=1.$$

This one is similar and also based on the formula (that we used there and which we invite you to remember)

$$\sum_{t=1}^{\infty}\frac{s!t!}{(s+t)!}=\frac{1}{s-1},$$

valid for any positive integer $s\geq 2$. Now we have

$$\begin{aligned}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{i!j!}{(i+j+2)!}&=\sum_{i=1}^{\infty}\frac{1}{(i+1)(i+2)}\sum_{j=1}^{\infty}\frac{(i+2)!j!}{(i+2+j)!}\\&=\sum_{i=1}^{\infty}\frac{1}{(i+1)(i+2)}\frac{1}{i+1}\\&=\sum_{i=1}^{\infty}\left(\frac{1}{(i+1)^2}-\frac{1}{(i+1)(i+2)}\right)\end{aligned}$$

$$= \sum_{i=1}^{\infty} \frac{1}{(i+1)^2} - \sum_{i=1}^{\infty} \frac{1}{(i+1)(i+2)} = \frac{\pi^2}{6} - 1 - \frac{1}{2} = \frac{\pi^2 - 9}{6}.$$

We used again the well-known telescoping sum

$$\sum_{i=1}^{\infty} \frac{1}{(i+1)(i+2)} = \sum_{i=1}^{\infty} \left(\frac{1}{i+1} - \frac{1}{i+2} \right) = \frac{1}{2},$$

and the result of the Basel problem (problem M38)

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}.$$

You can also prove that

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \frac{i!j!}{(i+j+3)!} = \frac{10 - \pi^2}{6},$$

can't you? Or, prove that

$$\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{i!j!}{(i+j+2)!} = \frac{\pi^2}{6}$$

(you will be needing a slight modification of the formula that we used).

H39. *Prove that*

$$\sum_{k=1}^{\infty} \frac{1}{k^4} = \frac{\pi^4}{90}.$$

Solution. We use again the equation

$$\binom{2n+1}{1} x^n - \binom{2n+1}{3} x^{n-1} + \cdots + (-1)^n \binom{2n+1}{2n+1} = 0$$

and its roots

$$\cot^2 \frac{k\pi}{2n+1}, \; k = 1, 2, \ldots, n$$

whose sum is, as we have seen in the solution of the Basel problem, $n(2n-1)/3$. We can also compute the sum of products of these roots taken two by two, which is

$$\sum_{1\le i<j\le n} \cot^2 \frac{i\pi}{2n+1} \cot^2 \frac{j\pi}{2n+1} = \frac{\binom{2n+1}{5}}{\binom{2n+1}{1}} = \frac{n(2n-1)(n-1)(2n-3)}{30}.$$

Therefore, the sum of the squares of these numbers is

$$\begin{aligned}\sum_{k=1}^{n} \cot^4 \frac{k\pi}{2n+1} &= \left(\frac{n(2n-1)}{3}\right)^2 - 2\frac{n(2n-1)(n-1)(2n-3)}{30} \\ &= \frac{n(2n-1)(4n^2+10n-9)}{45}.\end{aligned}$$

Further one we can also compute

$$\sum_{k=1}^{n} \frac{1}{\sin^4 \dfrac{k\pi}{2n+1}} = \sum_{k=1}^{n} \left(\cot^4 \frac{k\pi}{2n+1} + 2\cot^4 \frac{k\pi}{2n+1} + 1\right)$$

$$= \frac{n(2n-1)(4n^2+10n-9)}{45} + \frac{2n(2n-1)}{3} + n = \frac{8n(n+1)(n^2+n+3)}{45},$$

and one can use the same inequalities as before (changed just a little), namely

$$\cot^4 x < \frac{1}{x^4} < \frac{1}{\sin^4 x}, \ x \in \left(0, \frac{\pi}{2}\right).$$

to finally get

$$\frac{\pi^4}{90} \cdot \frac{2n(2n-1)(4n^2+10n-9)}{(2n+1)^4} < \sum_{k=1}^{n} \frac{1}{k^4} < \frac{\pi^4}{90} \cdot \frac{16n(n+1)(n^2+n+3)}{(2n+1)^4}$$

for all $n \geq 1$. Letting $n \to \infty$ yields the desired result.

This result was also proved for the first time by Euler.

H40. *Evaluate*

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \sum_{l=0}^{\infty} \frac{1}{k2^l + 1}.$$

Solution 1. We have

$$\begin{aligned}
\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \sum_{l=0}^{\infty} \frac{1}{k2^l+1} &= \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \sum_{l=0}^{\infty} \int_0^1 x^{k2^l} dx \\
&= \int_0^1 \sum_{l=0}^{\infty} \sum_{k=1}^{\infty} (-1)^{k-1} \frac{(x^{2^l})^k}{k} dx \\
&= \int_0^1 \sum_{l=0}^{\infty} \ln(1 + x^{2^l}) dx \\
&= \int_0^1 \ln\left(\prod_{l=0}^{\infty} (1 + x^{2^l})\right) dx \\
&= \int_0^1 \ln \frac{1}{1-x} dx = -\int_0^1 \ln(1-x) dx \\
&= [(1-x)\ln(1-x) + x]_0^1 = 1.
\end{aligned}$$

The infinite product is easy calculated from

$$\prod_{l=0}^{n} (1 + x^{2^l}) = \frac{1 - x^{2^{n+1}}}{1-x}$$

by passing to the limit for $n \to \infty$. Also the final evaluation needs the calculation of the *limit* of $(1-x)\ln(1-x)$ for $x \to 1$, which is easily shown to be 0, by using l'Hôpital's rule.

Solution 2. We again interchange the order of summation, based on the absolute convergence of the series – but in a different way. Namely we have

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \sum_{l=0}^{\infty} \frac{1}{k2^l+1} = \sum_{m=2}^{\infty} \frac{1}{m} \sum \frac{(-1)^{k-1}}{k}$$

where the inner sum is over those positive integers k for which there exists nonnegative integer l such that $m = k2^l + 1$. Now, if $m - 1 = 2^p q$ with nonnegative integer p and odd positive integer q, we see that l can be from 0 to p (for each such l the corresponding k is $2^{p-l}q$), and the above inner sum is

$$\frac{1}{q}\left(-\frac{1}{2^p} - \frac{1}{2^{p-1}} - \cdots - \frac{1}{2} + 1\right) = \frac{1}{2^p q} = \frac{1}{m-1}.$$

Thus the initial sum becomes

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \sum_{l=0}^{\infty} \frac{1}{k2^l + 1} = \sum_{m=2}^{\infty} \frac{1}{m} \frac{1}{m-1} = \sum_{m=2}^{\infty} \left(\frac{1}{m-1} - \frac{1}{m}\right) = 1$$

(and we finished our job with the same most often met telescoping sum). The problem was in the Putnam Competition in the year 2016.

H41. *Let $a_1, a_2, \ldots, a_{100}$ be nonnegative real numbers such that*

$$a_1^2 + a_2^2 + \cdots + a_{100}^2 = 1.$$

Prove that

$$a_1^2 a_2 + a_2^2 a_3 + \cdots + a_{100}^2 a_1 \le \frac{\sqrt{2}}{3}.$$

Solution. Let S be the sum from the left-hand side of the inequality. By the Cauchy-Schwarz inequality we have

$$\begin{aligned}(3S)^2 &= \left(\sum_{k=1}^{100} a_{k+1}(a_k^2 + 2a_{k+1}a_{k+2})\right)^2 \\ &\le \left(\sum_{k=1}^{100} a_{k+1}^2\right)\left(\sum_{k=1}^{100} (a_k^2 + 2a_{k+1}a_{k+2})^2\right).\end{aligned}$$

Please pay attention to the sums! We considered the indices modulo 100 (that is, $a_{101} = a_1$ and $a_{102} = a_2$), hence we have

$$\sum_{k=1}^{100} a_{k+1}^2 = \sum_{k=1}^{100} a_k^2 = 1,$$

so the above inequality reads

$$(3S)^2 \leq \sum_{k=1}^{100}(a_k^2 + 2a_{k+1}a_{k+2})^2.$$

We then have

$$\begin{aligned}\sum_{k=1}^{100}(a_k^2 + 2a_{k+1}a_{k+2})^2 &= \sum_{k=1}^{100}(a_k^4 + 4a_k^2a_{k+1}a_{k+2} + 4a_{k+1}^2a_{k+2}^2)\\ &\leq \sum_{k=1}^{100}(a_k^4 + 2a_k^2(a_{k+1}^2 + a_{k+2}^2) + 4a_{k+1}^2a_{k+2}^2)\\ &= \sum_{k=1}^{100}(a_k^4 + 6a_k^2a_{k+1}^2 + 2a_k^2a_{k+2}^2),\end{aligned}$$

where we used $2xy \leq x^2 + y^2$ for any real numbers x and y, and

$$\sum_{k=1}^{100} a_{k+1}^2a_{k+2}^2 = \sum_{k=1}^{100} a_k^2a_{k+1}^2.$$

Now we clearly have

$$\sum_{k=1}^{100}(a_k^4 + 2a_k^2a_{k+1}^2 + 2a_k^2a_{k+1}^2) \leq \left(\sum_{k=1}^{100} a_k^2\right)^2 = 1$$

and, also,

$$\begin{aligned}\sum_{k=1}^{100} 4a_k^2a_{k+1}^2 &\leq 4\left(\sum_{j=1}^{50} a_{2j-1}^2\right)\left(\sum_{j=1}^{50} a_{2j}^2\right) \leq \left(\sum_{j=1}^{50} a_{2j}^2 + \sum_{j=1}^{50} a_{2j}^2\right)^2\\ &= \left(\sum_{k=1}^{100} a_k^2\right)^2 = 1,\end{aligned}$$

because $4xy \le (x+y)^2$ holds for real numbers x and y.
Putting all these together we get

$$\begin{aligned}(3S)^2 &\le \sum_{k=1}^{100}(a_k^4 + 6a_k^2a_{k+1}^2 + 2a_k^2a_{k+2}^2)\\ &= \sum_{k=1}^{100}(a_k^4 + 2a_k^2a_{k+1}^2 + 2a_k^2a_{k+1}^2) + \sum_{k=1}^{100} 4a_k^2a_{k+1}^2\\ &\le 1+1=2,\end{aligned}$$

that is $S \le \sqrt{2}/3$, as desired.

H42. *Let $x_1, \ldots, x_{100}$ be nonnegative real numbers such that*

$$x_i + x_{i+1} + x_{i+2} \le 1 \textit{ for all } i = 1, \ldots, 100$$

(set $x_{101} = x_1, x_{102} = x_2$). Find the maximal possible value of the sum

$$S = \sum_{i=1}^{100} x_i x_{i+2}.$$

Solution. We have, by hypothesis

$$\begin{aligned}\sum_{i=1}^{100} x_i x_{i+2} &= \sum_{j=1}^{50}(x_{2j-1}x_{2j+1} + x_{2j}x_{2j+2})\\ &\le \sum_{j=1}^{50}((1 - x_{2j} - x_{2j+1})x_{2j+1} + x_{2j}(1 - x_{2j} - x_{2j+1}))\\ &= \sum_{j=1}^{50}(1 - x_{2j} - x_{2j+1})(x_{2j} + x_{2j+1}).\end{aligned}$$

But $xy \le ((x+y)/2)^2$ for every $x, y \in \mathbb{R}$, hence

$$(1 - x_{2j} - x_{2j+1})(x_{2j} + x_{2j+1}) \le \left(\frac{1 - x_{2j} - x_{2j+1} + x_{2j} + x_{2j+1}}{2}\right)^2 = \frac{1}{4},$$

for every $j = 1, 2, \ldots, 50$, and the sum S can be majorized as

$$S \le \sum_{j=1}^{50} \frac{1}{4} = \frac{25}{2}.$$

Moreover, the value $25/2$ is indeed assumed by S if we choose, for example, $x_{2j-1} = 1/2$ and $x_{2j} = 0$ for $j = 1, 2, \ldots, 50$. That is, the required maximum is $25/2$.

H43. *Prove that for any real numbers $x_1, x_2, \ldots, x_n$ and any nonnegative real numbers $r_1, r_2, \ldots, r_n$ the inequality*

$$\sum_{i,j=1}^{n} \min(r_i, r_j) x_i x_j \ge 0$$

holds. (The sum is over all pairs (i, j) with $1 \le i \le n$ and $1 \le j \le n$.)

Solution. Due to symmetry we can assume that $r_1 \le r_2 \le \cdots \le r_n$. Hence we can find nonnegative numbers s_k such that

$$r_i = \sum_{k=1}^{i} s_k$$

(explicitly, $s_1 = r_1$ and $s_k = r_k - r_{k-1}$ for $1 < k \le n$). Thus we can write

$$\begin{aligned}\sum_{i,j=1}^{n} \min(r_i, r_j) x_i x_j &= \sum_{i,j=1}^{n} r_{\min(i,j)} x_i x_j = \sum_{i,j=1}^{n} \sum_{k=1}^{\min(i,j)} s_k x_i x_j \\ &= \sum_{k=1}^{n} \sum_{i=k}^{n} \sum_{j=k}^{n} s_k x_i x_j = \sum_{k=1}^{n} s_k \left(\sum_{i=k}^{n} x_i \right)^2 \ge 0.\end{aligned}$$

H44. *Let $a_1, b_1, a_2, b_2, \ldots, a_n, b_n$ be nonnegative real numbers. Prove that*

$$\sum_{i,j=1}^{n} \min(a_i a_j, b_i b_j) \le \sum_{i,j=1}^{n} \min(a_i b_j, a_j b_i).$$

Solution. We will use the result of the previous problem for the numbers

$$r_i = \frac{\max(a_i, b_i)}{\min(a_i, b_i)} - 1$$

and

$$x_i = \operatorname{sgn}(a_i - b_i) \min(a_i, b_i)$$

for $1 \leq i \leq n$, with the observation that, if $\min(a_i, b_i) = 0$, then for r_i we can choose any nonnegative real number. Note that $\operatorname{sgn}(x)$ is $-1, 0$, or 1 as x is negative, 0, or positive, respectively. We then have

$$\sum_{i,j=1}^{n} \min(a_i b_j, a_j b_i) - \sum_{i,j=1}^{n} \min(a_i a_j, b_i b_j)$$
$$= \sum_{i,j=1}^{n} (\min(a_i b_j, a_j b_i) - \min(a_i a_j, b_i b_j))$$
$$= \sum_{i,j=1}^{n} x_i x_j \min(r_i, r_j) \geq 0,$$

according to the previous example.

Of course, we still need to prove that

$$\min(a_i b_j, a_j b_i) - \min(a_i a_j, b_i b_j) = x_i x_j \min(r_i, r_j)$$

for all $i, j \in \{1, 2, \ldots, n\}$.

Note first that if we have $\min(a_i, b_i) = 0$ or $\min(a_j, b_j) = 0$ the equality holds since both sides are zero (and it doesn't matter what we choose for r_i or r_j respectively). So we further assume that both $\min(a_i, b_i)$ and $\min(a_j, b_j)$ are nonzero, and we need to prove that

$$\min(a_i b_j, a_j b_i) - \min(a_i a_j, b_i b_j)$$
$$= \operatorname{sgn}(a_i - b_i) \min(a_i, b_i) \operatorname{sgn}(a_j - b_j) \min(a_j, b_j)$$
$$\cdot \min\left(\frac{\max(a_i, b_i)}{\min(a_i, b_i)} - 1, \frac{\max(a_j, b_j)}{\min(a_j, b_j)} - 1\right).$$

Note that if we interchange a_i and b_i, then both sides of this equation are multiplied by -1. Thus we may assume that $a_i \geq b_i$ and similarly we may assume $a_j \geq b_j$. In this case the right hand side simplifies to

$$b_i b_j \min\left(\frac{a_i}{b_i} - 1, \frac{a_j}{b_j} - 1\right) = b_i b_j \left(\min\left(\frac{a_i}{b_i}, \frac{a_j}{b_j}\right) - 1\right)$$

$$= \min(a_i b_j, a_j b_i) - b_i b_j = \min(a_i b_j, a_j b_i) - \min(a_i a_j, b_i b_j),$$

which is exactly what we wanted to prove.

Some related inequalities and refinements of the last two examples can be found in the article *On Some Elementary Inequalities* by Titu Andreescu and Gabriel Dospinescu (see it in *Mathematical Reflections: The First Two Years*).

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